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5

Laplace Transforms

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5.1 Introduction*

The Laplace transform has been introduced into the mathematical literature by a variety of procedures. Among these are: (a) in its relation to the Heaviside operational calculus, (b) as an extension of the Fourier integral, (c) by the selection of a particular form for the kernel in the general Integral transform, (d) by a direct definition of the Laplace transform, and (e) as a mathematical procedure that involves multiplying the function $f(t)$ by $e^{-st}\,dt$ and integrating over the limits 0 to ∞. We will adopt this latter procedure.

Not all functions $f(t)$, where t is any variable, are Laplace transformable. For a function $f(t)$ to be Laplace transformable, it must satisfy the Dirichlet conditions — a set of sufficient but not necessary conditions. These are

1. $f(t)$ must be piecewise continuous; that is, it must be single valued but can have a finite number of finite isolated discontinuities for $t > 0$.
2. $f(t)$ must be of exponential order; that is, $f(t)$ must remain less than $Me^{-a_o t}$ as t approaches ∞, where M is a positive constant and a_o is a real positive number.

For example, such functions as: $\tan \beta t$, $\cot \beta t$, e^{t^2} are not Laplace transformable. Given a function $f(t)$ that satisfies the Dirichlet conditions, then

$$F(s) = \int_0^\infty f(t) e^{-st} dt \quad \text{written } \mathcal{L}\{f(t)\} \tag{1.1}$$

is called the Laplace transformation of $f(t)$. Here s can be either a real variable or a complex quantity. Observe the shorthand notation $\mathcal{L}\{f(t)\}$ to denote the Laplace transformation of $f(t)$. Observe also that only ordinary integration is involved in this integral.

*All the contour integrations in the complex plane are counterclockwise.

To amplify the meaning of condition (2), we consider piecewise continuous functions, defined for all positive values of the variable t, for which

$$\lim_{t\to\infty} f(t)e^{-ct} = 0, \quad c = \text{real constant}.$$

Functions of this type are known as functions of exponential order. Functions occurring in the solution for the time response of stable linear systems are of exponential order zero. Now we can recall that the integral $\int_0^\infty f(t)\, e^{-st}\, dt$ converges if

$$\int_0^\infty \left| f(t)e^{-st} \right| dt < \infty, \quad s = \sigma + j\omega$$

If our function is of exponential order, we can write this integral as

$$\int_0^\infty \left| f(t) \right| e^{-ct} e^{-(\sigma-c)t}\, dt.$$

This shows that for σ in the range $\sigma > 0$ (σ is the abscissa of convergence) the integral converges; that is

$$\int_0^\infty \left| f(t)e^{-st} \right| dt < \infty, \quad \operatorname{Re}(s) > c.$$

The restriction in this equation, namely, $\operatorname{Re}(s) = c$, indicates that we must choose the path of integration in the complex plane as shown in Figure 5.1.

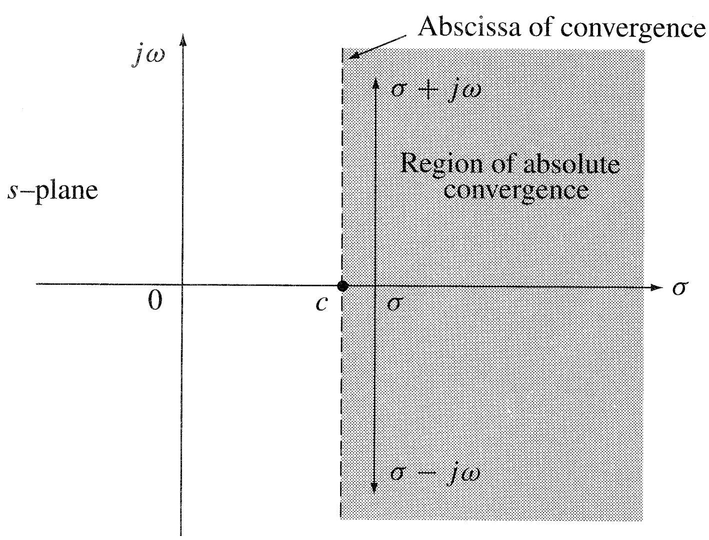


FIGURE 5.1 Path of integration for exponential order function.

5.2 Laplace Transform of Some Typical Functions

We illustrate the procedure in finding the Laplace transform of a given function $f(t)$. In all cases it is assumed that the function $f(t)$ satisfies the conditions of Laplace transformability.

Example 5.2.1

Find the Laplace transform of the unit step function $f(t) = u(t)$, where $u(t) = 1$, $t > 0$, $u(t) = 0$, $t < 0$.

Solution

By (1.1) we write

$$\mathscr{L}\{u(t)\} = \int_0^\infty u(t)e^{-st}dt = \int_0^\infty e^{-st}dt = \frac{e^{-st}}{s}\Bigg|_0^\infty = \frac{1}{s}. \tag{2.1}$$

The region of convergence is found from the expression $\int_0^\infty |e^{-st}|\,dt = \int_0^\infty e^{-\sigma t}\,dt < \infty$, which is the entire right half-plane, $\sigma > 0$.

Example 5.2.2

Find the Laplace transform of the function $f(t) = 2\sqrt{\frac{t}{\pi}}$

$$F(s) = \frac{2}{\sqrt{\pi}}\int_0^\infty t^{\frac{1}{2}}e^{-st}dt. \tag{2.2}$$

To carry out the integration, define the quantity $x = t^{\frac{1}{2}}$, then $dx = \frac{1}{2}t^{-\frac{1}{2}}\,dt$, from which $dt = 2t^{\frac{1}{2}}\,dx = 2x\,dx$. Then

$$F(s) = \frac{4}{\sqrt{\pi}}\int_0^\infty x^2e^{-sx^2}dx.$$

But the integral

$$\int_0^\infty x^2e^{-sx^2}dx = \frac{\sqrt{\pi}}{4s^{3/2}}.$$

Thus, finally,

$$F(s) = \frac{1}{s^{3/2}}. \tag{2.3}$$

Example 5.2.3

Find the Laplace transform of $f(t) = \text{erfc}\,\dfrac{k}{2\sqrt{t}}$, where the error function, erf t, and the complementary error function, erfc t, are defined by

$$\text{erf}\, t = \frac{2}{\sqrt{\pi}} \int_o^t e^{-u^2} du, \quad \text{erfc}\, t = \frac{2}{\sqrt{\pi}} \int_t^\infty e^{-u^2} du,$$

Solution

Consider the integral

$$I = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-st} \left[\int_{\frac{\lambda}{\sqrt{t}}}^\infty e^{-u^2} du \right] dt \quad \text{where } \lambda = \frac{k}{2}. \tag{2.4}$$

Change the order of integration, noting that $u = \dfrac{\lambda}{\sqrt{t}}$, $t = \dfrac{\lambda^2}{u^2}$

$$I = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-u^2} \left[\int_{\frac{\lambda^2}{u^2}}^\infty e^{-st} \right] dt\, du = \frac{2}{s\sqrt{\pi}} \int_0^\infty \exp\left(-u^2 - \frac{\lambda^2 s}{u^2} \right) du$$

The value of this integral is known

$$= \frac{2}{s\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{2} e^{-2\lambda\sqrt{s}},$$

which leads to

$$\mathcal{L}\left\{ \text{erfc}\, \frac{k}{2\sqrt{t}} \right\} = \frac{1}{s} \exp\left\{ -k\sqrt{s} \right\}. \tag{2.5}$$

Example 5.2.4

Find the Laplace transform of the function $f(t) = \sinh at$.

Solution

Express the function sinh at in its exponential form

$$\sinh at = \frac{e^{at} - e^{-at}}{2}.$$

The Laplace transform becomes

$$\mathscr{L}\{\sinh at\} = \frac{1}{2}\int_0^\infty \left[e^{-(s-a)t} - e^{-(s+a)t}\right]dt$$
$$= \frac{a}{s^2 - a^2}\,. \tag{2.6}$$

A moderate listing of functions $f(t)$ and their Laplace transforms $F(s) = \mathscr{L}\{f(t)\}$ are given in Table 5.1, in the Appendix.

5.3 Properties of the Laplace Transform

We now develop a number of useful properties of the Laplace transform; these follow directly from (1.1). Important in developing certain properties is the definition of $f(t)$ at $t = 0$, a quantity written $f(0+)$ to denote the limit of $f(t)$ as t approaches zero, assumed from the positive direction. This designation is consistent with the choice of function response for $t > 0$. This means that $f(0+)$ denotes the initial condition. Correspondingly, $f^{(n)}(0+)$ denotes the value of the nth derivative at time $t = 0+$, and $f^{(-n)}(0+)$ denotes the nth time integral at time $t = 0+$. This means that the direct Laplace transform can be written

$$F(s) = \lim_{\substack{R\to\infty \\ a\to 0+}} \int_a^R f(t)e^{-st}dt, \quad R > 0, a > 0\,. \tag{3.1}$$

We proceed with a number of theorems.

Theorem 5.3.1 Linearity
The Laplace transform of the linear sum of two Laplace transformable functions $f(t) + g(t)$ with respective abscissas of convergence σ_f and σ_g, with $\sigma_g > \sigma_f$, is

$$\mathscr{L}\{f(t) + g(t)\} = F(s) + G(s)\,. \tag{3.2}$$

Proof
From (3.1) we write

$$\mathscr{L}\{f(t) + g(t)\} = \int_0^\infty [f(t) + g(t)]e^{-st}dt = \int_0^\infty f(t)e^{-st}dt + \int_0^\infty g(t)e^{-st}dt,$$
$$\operatorname{Re}(s) > \sigma_g\,.$$

Thus,

$$\mathscr{L}\{f(t) + g(t)\} = F(s) + G(s)\,.$$

As a direct extension of this result, for K_1 and K_2 constants,

$$\mathscr{L}\{K_1 f(t) + K_2 g(t)\} = K_1 F(s) + K_2 G(s)\,. \tag{3.3}$$

Theorem 5.3.2 Differentiation
Let the function $f(t)$ be piecewise continuous with sectionally continuous derivatives $df(t)/dt$ in every interval $0 \le t \le T$. Also let $f(t)$ be of exponential order e^{ct} as $t \to \infty$. Then when $\operatorname{Re}(s) > c$, the transform of $df(t)/dt$ exists and

$$\mathcal{L}\left\{\frac{df(t)}{dt}\right\} = s\,\mathcal{L}\{f(t)\} - f(0+) = sF(s) - f(0+). \tag{3.4}$$

Proof
Begin with (3.1) and write

$$\mathcal{L}\left\{\frac{df(t)}{dt}\right\} = \lim_{T\to\infty}\int_0^T \frac{df(t)}{dt} e^{-st}\,dt\,.$$

Write the integral as the sum of integrals in each interval in which the integrand is continuous. Thus, we write

$$\int_0^T e^{-st} f^{(1)}(t)\,dt = \int_0^{t_1}[\;] + \int_{t_1}^{t_2}[\;] + \cdots + \int_{t_{n-1}}^{T}[\;].$$

Each of these integrals is integrated by parts by writing

$$\begin{aligned} u &= e^{-st} & du &= -se^{-st}dt \\ dv &= \frac{df}{dt}dt & v &= f \end{aligned}$$

with the result

$$e^{-st} f(t)\Big|_0^{t_1} + e^{-st} f(t)\Big|_{t_1}^{t_2} + \cdots + e^{-st} f(t)\Big|_{t_{n-1}}^{T} + s\int_0^T e^{-st} f(t)\,dt\,.$$

But $f(t)$ is continuous so that $f(t_1 - 0) = f(t_1 + 0)$, and so forth, hence

$$\int_0^T e^{-st} f^{(1)}(t)\,dt = -f(0+) + e^{-sT} f(T) + s\int_0^T e^{-st} f(t)\,dt\,.$$

However, with $\lim_{t\to\infty} f(t)e^{-st} = 0$ (otherwise the transform would not exist), then the theorem is established.

Theorem 5.3.3 Differentiation
Let the function $f(t)$ be piecewise continuous, have a continuous derivative $f^{(n-1)}(t)$ of order $n - 1$ and a sectionally continuous derivative $f^{(n)}(t)$ in every finite interval $0 \le t \le T$. Also, let $f(t)$ and all its derivatives through $f^{(n-1)}(t)$ be of exponential order e^{ct} as $t \to \infty$. Then the transform of $f^{(n)}(t)$ exists when $\mathrm{Re}(s) > c$ and it has the following form:

$$\mathcal{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1} f(0+) - s^{n-2} f^{(1)}(0+) - \cdots - s^{n-n} f^{(n-1)}(0+)\,. \tag{3.5}$$

Proof
The proof follows as a direct extension of the proof of Theorem 5.3.2.

Example 5.3.1

Find $\mathcal{L}\{t^m\}$ where m is any positive integer.

Solution
The function $f(t) = t^m$ satisfies all the conditions of Theorem 5.3.3 for any positive c. Thus,

$$f(0+) = f^{(1)}(0+) = \cdots = f^{(m-1)}(0+) = 0$$

$$f^{(m)}(t) = m!, \quad f^{(m+1)}(t) = 0 .$$

By (3.5) with $n = m + 1$ we have

$$\mathcal{L}\{f^{(m+1)}(t)\} = 0 = s^{m+1}\mathcal{L}\{t^m\} - m! .$$

It follows, therefore, that

$$\mathcal{L}\left\{t^m\right\} = \frac{m!}{s^{m+1}} .$$

Theorem 5.3.4 Integration

If $f(t)$ is sectionally continuous and has a Laplace transform, then the function $\int_o^t f(\xi)\,d\xi$ has the Laplace transform given by

$$\mathcal{L}\left\{\int_0^t f(\xi)d\xi\right\} = \frac{F(s)}{s} + \frac{1}{s} f^{(-1)}(0+). \tag{3.6}$$

Proof
Because $f(t)$ is Laplace transformable, its integral is written

$$\mathcal{L}\left\{\int_{-\infty}^t f(\xi)d\xi\right\} = \int_0^\infty \left[\int_{-\infty}^t f(\xi)d\xi\right] e^{-st}dt .$$

This is integrated by parts by writing

$$u = \int_{-\infty}^t f(\xi)d\xi \qquad du = f(\xi)d\xi = f(t)dt$$

$$dv = e^{-st}dt \qquad v = -\frac{1}{s}e^{-st} .$$

Then

$$\mathcal{L}\left\{\int_{-\infty}^t f(\xi)d\xi\right\} = \left[-\frac{e^{-st}}{s}\int_{-\infty}^t f(\xi)d\xi\right]\Bigg|_0^\infty + \frac{1}{s}\int_0^\infty f(t)e^{-st}dt$$

$$= \frac{1}{s}\int_0^\infty f(t)e^{-st}dt + \frac{1}{s}\int_{-\infty}^0 f(\xi)d\xi$$

from which

$$\mathcal{L}\left\{\int_0^t f(\xi)\,d\xi\right\} = \frac{1}{s}F(s) + \frac{1}{s}f^{(-1)}(0+)$$

where $[f^{(-1)}(0+)/s]$ is the initial value of the integral of $f(t)$ at $t = 0+$. The negative number in the bracketed exponent indicates integration.

Example 5.3.2

Deduce the value of $\mathcal{L}\{\sin at\}$ from $\mathcal{L}\{\cos at\}$ by employing Theorem 5.3.4.

Solution

By ordinary integration

$$\int_0^t \cos ax\,dx = \frac{\sin at}{a}.$$

From Theorem 5.3.4 we can write, knowing that $\mathcal{L}\{\cos at\} = \dfrac{s}{s^2 + a^2}$.

$$\mathcal{L}\left\{\frac{\sin at}{a}\right\} = \frac{1}{s^2 + a^2}$$

so that

$$\mathcal{L}\{\sin at\} = \frac{a}{s^2 + a^2}.$$

Theorem 5.3.5

Division of the transform of a function by s corresponds to integration of the function between the limits 0 and t

$$\mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(\xi)\,d\xi$$
$$\mathcal{L}^{-1}\left\{\frac{F(s)}{s^2}\right\} = \int_0^t \int_0^\xi f(\lambda)\,d\lambda\,d\xi \qquad (3.7)$$

and so forth, for division by s^n, provided that $f(t)$ is Laplace transformable.

Proof

The proof of this theorem follows from Theorem 5.3.4.

Theorem 5.3.6 Multiplication by t

If $f(t)$ is piecewise continuous and of exponential order, then each of the Laplace transforms: $\mathcal{L}\{f(t)$, $\mathcal{L}\{tf(t)$, $\mathcal{L}\{t^2f(t)$,…is uniformly convergent with respect to s when $s = c$, where $\sigma > c$, and

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n F(s)}{ds^n}. \qquad (3.8)$$

Further

$$\lim_{s\to\infty}\frac{d^nF(s)}{ds^n}=0\,,\quad \mathscr{L}_{s\to\infty}\left\{t^nf(t)\right\}=0\,,\quad n=1,2,3,\ldots$$

Proof
It follows from (3.1) when this integral is uniformly convergent and the integral converges, that

$$\frac{\partial F(s)}{\partial s}=\int_0^\infty e^{-st}(-t)f(t)\,dt=\mathscr{L}\left\{-tf(t)\right\}.$$

Further, it follows that

$$\frac{\partial^2 F(s)}{\partial s^2}=\int_0^\infty e^{-st}(-t)^2f(t)\,dt=\mathscr{L}\left\{t^2f(t)\right\}.$$

Similar procedures follow for derivatives of higher order.

Theorem 5.3.7 Differentiation of a Transform
Differentiation of the transform of a function $f(t)$ corresponds to the multiplication of the function by $-t$; thus

$$\frac{d^nF(s)}{ds^n}=F^{(n)}(s)=\mathscr{L}\left\{(-t)^nf(t)\right\},\quad n=1,2,3,\ldots.\tag{3.9}$$

Proof
This is a restatement of Theorem 5.3.6. This theorem is often useful for evaluating some types of integrals, and can be used to extend the table of transforms.

Example 5.3.3

Employ Theorem 5.3.7 to evaluate $\partial F(s)/\partial s$ for the function $f(t) = \sinh at$.

Solution
Initially we establish sinh at

$$\mathscr{L}\left\{\sinh at\right\}=\int_0^\infty e^{-st}\left[\frac{e^{at}-e^{-at}}{2}\right]dt=\frac{a}{s^2-a^2}=F(s).$$

By Theorem 5.3.7

$$\frac{\partial F(s)}{\partial s}=\int_0^\infty(-t)\sinh at\,e^{-st}\,dt=\frac{\partial}{\partial s}\left[\frac{a}{s^2-a^2}\right]=-\frac{2as}{\left(s^2-a^2\right)^2}$$

from which

$$\int_0^\infty e^{-st}t\sinh at\,dt=\mathscr{L}\left\{t\sinh at\right\}=\frac{2as}{\left(s^2-a^2\right)^2}.$$

We can, of course, differentiate $F(s)$ with respect to a. In this case, Theorem 5.3.7 does not apply. However, the result is significant and is

$$\frac{\partial F(s)}{\partial a} = \int_0^{\infty} e^{-st}(t\cosh at)\,dt = \mathscr{L}\{t\cosh at\} = \frac{\partial}{\partial a}\left[\frac{a}{s^2-a^2}\right] = \frac{s^2+a^2}{\left(s^2-a^2\right)^2}.$$

Theorem 5.3.8 Complex Integration

If $f(t)$ is Laplace transformable and provided that $\lim_{t\to 0+} \dfrac{f(t)}{t}$ exists, the integral of the function $\int_s^{\infty} F(s)\, ds$ corresponds to the Laplace transform of the division of the function $f(t)$ by t,

$$\mathscr{L}\left\{\frac{f(t)}{t}\right\} = \int_0^{\infty} F(s)\,ds. \qquad (3.10)$$

Proof

Let $F(s)$ be piecewise continuous in each finite interval and of exponential order. Then

$$F(s) = \int_0^{\infty} e^{-st} f(t)\,dt$$

is uniformly convergent with respect to s. Consequently, we can write for $\mathrm{Re}(s) > c$ and any $a > c$

$$\int_s^{a} F(s)\,ds = \int_s^{a}\int_0^{\infty} e^{-st} f(t)\,dt\,ds.$$

Express this in the form

$$= \int_0^{\infty} f(t)\int_s^{a} e^{-st}\,ds\,dt = \int_0^{\infty} \frac{f(t)}{t}\left(e^{-st} - e^{-at}\right)dt.$$

Now if $f(t)/t$ has a limit as $t \to 0$, then the latter function is piecewise continuous and of exponential order. Therefore, the last integral is uniformly convergent with respect to a. Thus, as a tends to infinity

$$\int_s^{\infty} F(s)\,ds = \mathscr{L}\left\{\frac{f(t)}{t}\right\}.$$

Theorem 5.3.9 Time Delay; Real Translation

The substitution of $t - \lambda$ for the variable t in the transform $\mathscr{L}\{f(t)\}$ corresponds to the multiplication of the function $F(s)$ by $e^{-\lambda s}$; that is,

$$\mathscr{L}\{f(t-\lambda)\} = e^{-s\lambda} F(s) \;. \qquad (3.11)$$

Proof

Refer to Figure 5.2, which shows a function $f(t)\,u(t)$ and the same function delayed by the time $t = \lambda$, where λ is a positive constant.

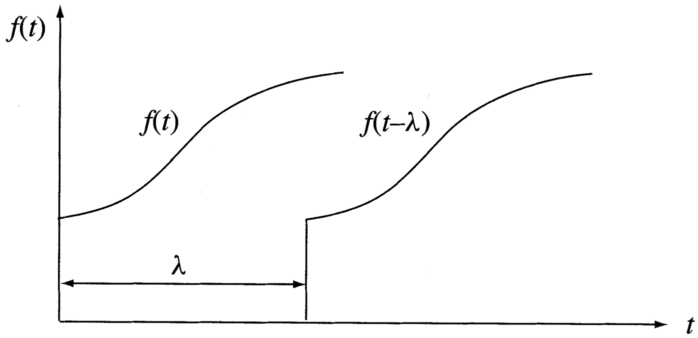


FIGURE 5.2 A function $f(t)$ at the time $t = 0$ and delayed time $t = \lambda$.

We write directly

$$\mathcal{L}\{f(t-\lambda)u(t-\lambda)\} = \int_0^\infty f(t-\lambda)u(t-\lambda)e^{-st}dt.$$

Now introduce a new variable $\tau = t - \lambda$. This converts this equation to the form

$$\mathcal{L}\{f(\tau)u(\tau)\} = e^{-s\lambda}\int_{-\lambda}^\infty f(\tau)u(\tau)e^{-s\tau}d\tau = e^{-s\lambda}\int_0^\infty f(\tau)e^{-s\tau}d\tau = e^{-s\lambda}F(s)$$

because $u(\tau) = 0$ for $-\lambda \le t \le 0$.

We would similarly find that

$$\mathcal{L}\{f(t+\lambda)u(t+\lambda)\} = e^{s\lambda}F(s). \tag{3.12}$$

Example 5.3.4

Find the Laplace transform of the pulse function shown in Figure 5.3.

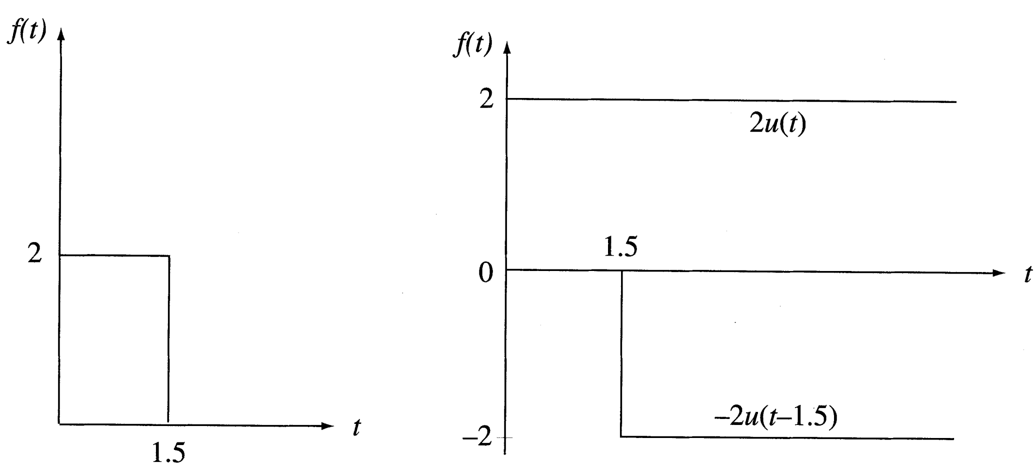


FIGURE 5.3 Pulse function and its equivalent representation.

Solution

Because the pulse function can be decomposed into step functions, as shown in Figure 5.3, its Laplace transform is given by

$$\mathscr{L}\{2[u(t)-u(t-1.5)]\}=2\left[\frac{1}{s}-\frac{1}{s}e^{-1.5s}\right]=\frac{2}{s}\left(1-e^{-1.5s}\right)$$

where the translation property has been used.

Theorem 5.3.10 Complex Translation

The substitution of $s + a$ for s, where a is a real or complex, in the function $F(s + a)$, corresponds to the Laplace transform of the product $e^{-at}f(t)$.

Proof

We write

$$\int_0^\infty e^{-at}f(t)e^{-st}dt=\int_0^\infty f(t)e^{-(s+a)t}dt \quad \text{for } \mathrm{Re}(s)>c-\mathrm{Re}(a),$$

which is

$$F(s+a)=\mathscr{L}\{e^{-at}f(t)\}\,. \tag{3.13}$$

In a similar way we find

$$F(s-a)=\mathscr{L}\{e^{at}f(t)\}\,. \tag{3.14}$$

Theorem 5.3.11 Convolution

The multiplication of the transforms of two sectionally continuous functions $f_1(t)$ $(= F_1(s))$ and $f_2(t)$ $(= F_2(s))$ corresponds to the Laplace transform of the convolution of $f_1(t)$ and $f_2(t)$.

$$F_1(s)\,F_2(s)=\mathscr{L}\{f_1(t)*f_2(t)\} \tag{3.15}$$

where the asterisk $*$ is the shorthand designation for convolution.

Proof

By definition, the convolution of two functions $f_1(t)$ and $f_2(t)$ is

$$f_1(t)*f_2(t)=\int_0^\infty f_1(t-\tau)f_2(\tau)d\tau=\int_0^\infty f_1(\tau)f_2(t-\tau)d\tau\,. \tag{3.16}$$

Thus,

$$\mathscr{L}\{f_1(t)*f_s(t)\}=\mathscr{L}\left\{\int_0^\infty f_1(t-\tau)f_2(\tau)d\tau\right\}$$

$$=\int_0^\infty\left[\int_0^\infty f_1(t-\tau)f_2(\tau)d\tau\right]e^{-st}dt$$

$$=\int_0^\infty f_2(\tau)d\tau\int_0^\infty f_1(t-\tau)e^{-st}dt\,.$$

Now effect a change of variable, writing $t - \tau = \xi$ and therefore $dt = d\xi$, then

$$=\int_0^\infty f_2(\tau)d\tau\int_{-\tau}^\infty f_1(\xi)e^{-s(\xi+\tau)}d\xi\,.$$

But for positive time functions $f_1(\xi) = 0$ for $\xi < 0$, which permits changing the lower limit of the second integral to zero, and so

$$= \int_0^\infty f_2(\tau) e^{-s\tau} d\tau \int_0^\infty f_1(\xi) e^{-s\xi} d\xi,$$

which is

$$\mathcal{L}\{f_1(t) * f_2(t)\} = F_1(s) F_2(s).$$

Example 5.3.5

Given $f_1(t) = t$ and $f_2(t) = e^{at}$, deduce the Laplace transform of the convolution $t * e^{at}$ by the use of Theorem 5.3.11.

Solution

Begin with the convolution

$$t * e^{at} = \int_0^t (t-\tau) e^{a\tau} d\tau = \left. \frac{t e^{a\tau}}{a} \right|_0^t - \left[\frac{\tau e^{a\tau}}{a} - \frac{e^{a\tau}}{a^2} \right]_0^t = \frac{1}{a^2}\left(e^{at} - at - 1\right).$$

Then

$$\mathcal{L}\{t * e^{at}\} = \frac{1}{a^2}\left(\frac{1}{s-a} - \frac{1}{s^2} - \frac{1}{s}\right) = \frac{1}{s^2} \frac{1}{(s-a)}.$$

By Theorem 5.3.11 we have

$$F_1(s) = \mathcal{L}\{f_1(t)\} = \mathcal{L}\{t\} = \frac{1}{s^2}, \quad F_2(s) = \mathcal{L}\{f_2(t)\} = \mathcal{L}\{e^{at}\} = \frac{1}{s-a}.$$

and

$$\mathcal{L}\{t * e^{at}\} = \frac{1}{s^2} \frac{1}{(s-a)}.$$

Theorem 5.3.12

The multiplication of the transforms of three sectionally continuous functions $f_1(t)$, $f_2(t)$, and $f_3(t)$ corresponds to the Laplace transform of the convolution of the three functions

$$\mathcal{L}\{f_1(t) * f_2(t) * f_3(t)\} = F_1(s) F_2(s) F_3(s). \tag{3.17}$$

Proof

This is an extension of Theorem 5.3.11. The result is obvious if we write

$$F_1(s) F_2(s) F_3(s) = \mathcal{L}\{ f_1(t) * \mathcal{L}^{-1}\{ F_2(s) F_3(s)\}\}.$$

Example 5.3.6

Deduce the values of the convolution products: $1 * f(t)$; $1 * 1 * f(t)$.

Solution

By equations (3.14) and (3.16) we write directly

(a) For $f_1(t) = 1, f_2(t) = f(t), \mathscr{L}\{1 * f(t)\} = \frac{F(s)}{s} = \mathscr{L}\left\{\int_0^t f(\xi)\,d\xi\right\}$ by equation (3.7)

(b) For $f_1(t) = 1, f_2(t) = 1, f_3(t) = f(t), \mathscr{L}\{1 * 1 * f(t)\} = \frac{F(s)}{s^2} = \mathscr{L}\left\{\int_0^t \int_0^\xi f(\lambda)\,d\lambda\, d\xi\right\}$

Theorem 5.3.13 Frequency Convolution — s-plane

The Laplace transform of the product of two piecewise and sectionally continuous functions $f_1(t)$ and $f_2(t)$ corresponds to the convolution of their transforms, with

$$\mathscr{L}\{f_1(t)f_2(t)\} = \frac{1}{2\pi j}\left[F_1(s) * F_2(s)\right]. \tag{3.18}$$

Proof

Begin by considering the following line integral in the z-plane:

$$f_2(t) = \frac{1}{2\pi j}\int_{C_2} F_2(z)e^{zt}dz, \quad \sigma_2 = \text{axis of convergence}.$$

This means that the contour intersects the x-axis at $x_1 > \sigma_2$ (see Figure 5.4). Then we have

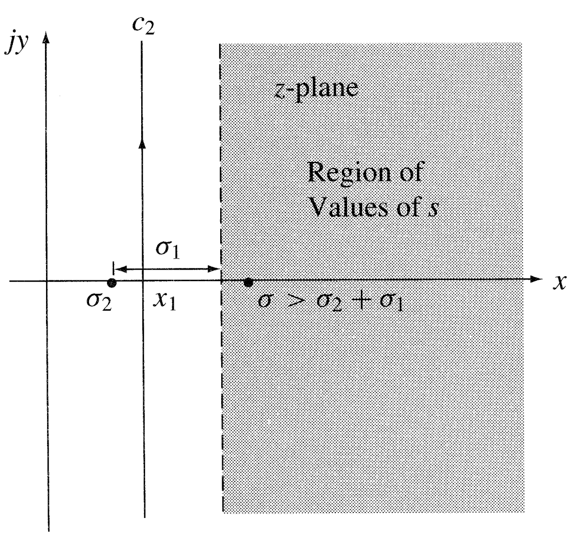


FIGURE 5.4 The contour C_2 and the allowed range of s.

$$\int_0^\infty f_1(t)f_2(t)e^{-st}dt = \frac{1}{2\pi j}\int_0^\infty f_1(t)\,dt\int_{C_2} F_2(z)e^{(z-s)t}dz.$$

Assume that the integral of $F_2(z)$ is convergent over the path of integration. This equation is now written in the form

$$\int_0^{\infty} f_1(t) f_2(t) e^{-st} dt = \frac{1}{2\pi j} \int_{\sigma_2 - j\infty}^{\sigma_2 + j\infty} F_2(z) dz \int_0^{\infty} f_1(t) e^{-(s-z)t} dt$$

$$= \frac{1}{2\pi j} \int_{\sigma_2 - j\infty}^{\sigma_2 + j\infty} F_2(z) F_1(s-z) dz \triangleq \mathscr{L}\{f_1(t) f_2(t)\}. \qquad (3.19)$$

The Laplace transform of $f_1(t)$, the integral on the right, converges in the range $\text{Re}(s - z) > \sigma_1$, where σ_1 is the abscissa of convergence of $f_1(t)$. In addition, $\text{Re}(z) = \sigma_2$ for the z-plane integration involved in (3.18). Thus, the abscissa of convergence of $f_1(t)\, f_2(t)$ is specified by

$$\text{Re}(s) > \sigma_1 + \sigma_2\,. \qquad (3.20)$$

This situation is portrayed graphically in Figure 5.4 for the case when both σ_1 and σ_2 are positive. As far as the integration in the complex plane is concerned, the semicircle can be closed either to the left or to the right just so long as $F_1(s)$ and $F_2(s)$ go to zero as $s \to \infty$.

Based on the foregoing, we observe the following:

- Poles of $F_1(s - z)$ are contained in the region $\text{Re}(s - z) < \sigma_1$
- Poles of $F_2(z)$ are contained in the region $\text{Re}(z) < \sigma_2$
- From (a) and (3.20) $\text{Re}(z) > \text{Re}(s - \sigma_1) > \sigma_2$
- Poles of $F_1(s - z)$ lie to the right of the path of integration
- Poles of $F_2(z)$ are to the left of the path of integration
- Poles of $F_1(s - z)$ are functions of s whereas poles of $F_2(z)$ are fixed in relation to s

Example 5.3.7

Find the Laplace transform of the function $f(t) = f_1(t)\, f_2(t) = e^{-t}\, e^{-2t}\, u(t)$.

Solution

From Theorem 5.3.13 and the absolute convergence region for each function, we have

$$F_1(s) = \frac{1}{s+1}, \quad \sigma_1 > -1$$

$$F_2(s) = \frac{1}{s+2}, \quad \sigma_2 > -2\,.$$

Further, $f(t) = \exp[-(2 + 1)t]\; u(t)$ implies that $\sigma_f = \sigma_1 + \sigma_2 = 3$. We now write

$$F_2(z) F_1(s-z) = \frac{1}{z+2} \frac{1}{s-z+1} = \frac{1}{3+s} \frac{1}{z-(1+s)} - \frac{1}{3+s} \frac{1}{z+2}.$$

To carry out the integration dictated by equation (3.19) we use the contour shown in Figure 5.5. If we select contour C_1 and use the residue theorem, we obtain

$$F(s) = \frac{1}{2\pi j} \oint_{C_1} F_2(z) F_1(s-z) dz = 2\pi j \,\text{Res}_s\left[F_2(z) F_1(s-z)\right]\Big|_{z=-2} = \frac{1}{s+3}.$$

The inverse of this transform is $\exp(-3t)$. If we had selected contour C_2, the residue theorem gives

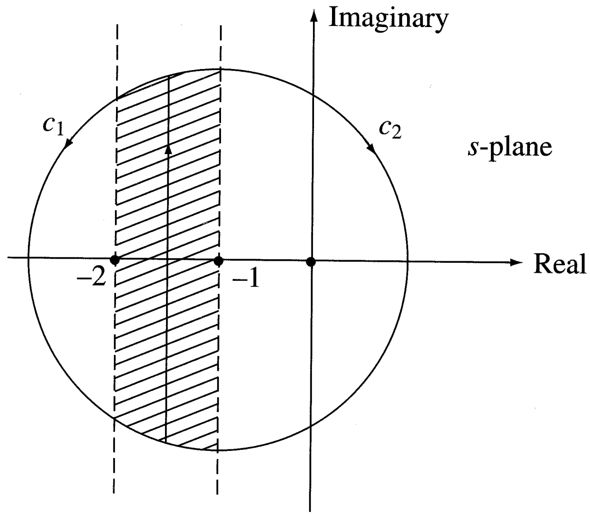


FIGURE 5.5 The contour for Example 5.3.7.

$$F(s) = \frac{1}{2\pi j}\oint_{C_2} F_2(z)F_1(s-z)\,dz = -2\pi j \operatorname{Res}\left[F_2(z)F_1(s-z)\right]\Big|_{z=1+s}$$

$$= -\left[-\frac{1}{s+3}\right] = \frac{1}{s+3}.$$

The inverse transform of this is also exp(–3t), as to be expected.

Theorem 5.3.14 Initial Value Theorem

Let $f(t)$ and $f^{(1)}(t)$ be Laplace transformable functions, then for case when lim $sF(s)$ as $s \to \infty$ exists,

$$\lim_{s\to\infty} sF(s) = \lim_{t\to 0+} f(t). \qquad (3.21)$$

Proof

Begin with equation (3.6) and consider

$$\lim_{s\to\infty}\int_0^{\infty}\frac{df}{dt}e^{-st}dt = \lim_{s\to\infty}\left[sF(s) - f(0+)\right].$$

Because $f(0+)$ is independent of s, and because the integral vanishes for $s \to \infty$, then

$$\lim_{s\to\infty}\left[sF(s) - f(0+)\right] = 0.$$

Furthermore, $f(0+) = \lim_{t\to 0+} f(t)$ so that

$$\lim_{s\to\infty} sF(s) = \lim_{t\to 0+} f(t).$$

If $f(t)$ has a discontinuity at the origin, this expression specifies the value of the impulse $f(0+)$. If $f(t)$ contains an impulse term, then the left-hand side does not exist, and the initial value property does not exist.

Theorem 5.3.15 Final Value Theorem
Let $f(t)$ and $f^{(1)}(t)$ be Laplace transformable functions, then for $t \to \infty$

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s). \tag{3.22}$$

Proof
Begin with equation (3.6) and Let $s \to 0$. Thus, the expression

$$\lim_{s\to 0}\int_0^{\infty} \frac{df}{dt} e^{-st} dt = \lim_{s\to 0}\left[sF(s) - f(0+)\right].$$

Consider the quantity on the left. Because s and t are independent and because $e^{-st} \to 1$ as $s \to 0$, then the integral on the left becomes, in the limit

$$\int_0^{\infty} \frac{df}{dt} dt = \lim_{t\to\infty} f(t) - f(0+).$$

Combine the latter two equations to get

$$\lim_{t\to\infty} f(t) - f(0+) = \lim_{s\to\infty} sF(s) - f(0+).$$

It follows from this that the final value of $f(t)$ is given by

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s).$$

This result applies $F(s)$ possesses a simple pole at the origin, but it does not apply if $F(s)$ has imaginary axis poles, poles in the right half plane, or higher order poles at the origin.

Example 5.3.8

Apply the final value theorem to the following two functions:

$$F_1(s) = \frac{s+a}{(s+a)^2 + b^2}, \quad F_2(s) = \frac{s}{s^2 + b^2}.$$

Solution
For the first function from $sF_1(s)$,

$$\lim_{s\to 0} \frac{s(s+a)}{(s+a)^2 + b^2} = 0.$$

For the second function,

$$sF(s) = \frac{s^2}{s^2 + b^2}.$$

However, this function has singularities on the imaginary axis at $s = \pm jb$, and the Final Value Theorem does not apply.

The important properties of the Laplace transform are contained in Table 5.2 in the Appendix.

5.4 The Inverse Laplace Transform

We employ the symbol $\mathscr{L}^{-1}\{F(s)\}$, corresponding to the direct Laplace transform defined in (1.1), to denote a function $f(t)$ whose Laplace transform is $F(s)$. Thus, we have the Laplace pair

$$F(s) = \mathscr{L}\{f(t)\}, \quad f(t) = \mathscr{L}^{-1}\{F(s)\}\,. \tag{4.1}$$

This correspondence between $F(s)$ and $f(t)$ is called the inverse Laplace transformation of $f(t)$.

Reference to Table 5.1 shows that $F(s)$ is a rational function in s if $f(t)$ is a polynomial or a sum of exponentials. Further, it appears that the product of a polynomial and an exponential might also yield a rational $F(s)$. If the square root of t appears on $f(t)$, we do not get a rational function in s. Note also that a continuous function $f(t)$ may not have a continuous inverse transform.

Observe that the $F(s)$ functions have been uniquely determined for the given $f(t)$ function by (1.1). A logical question is whether a given time function in Table 5.1 is the only t-function that will give the corresponding $F(s)$. Clearly, Table 5.1 is more useful if there is a unique $f(t)$ for each $F(s)$. This is an important consideration because the solution of practical problems usually provides a known $F(s)$ from which $f(t)$ must be found. This uniqueness condition can be established using the inversion integral. This means that there is a one-to-one correspondence between the direct and the inverse transform. This means that if a given problem yields a function $F(s)$, the corresponding $f(t)$ from Table 5.1 is the unique result. In the event that the available tables do not include a given $F(s)$, we would seek to resolve the given $F(s)$ into forms that are listed in Table 5.1. This resolution of $F(s)$ is often accomplished in terms of a partial fraction expansion.

A few examples will show the use of the partial fraction form in deducing the $f(t)$ for a given $F(s)$.

Example 5.4.1

Find the inverse Laplace transform of the function

$$F(s) = \frac{s-3}{s^2+5s+6}\,. \tag{4.2}$$

Solution

Observe that the denominator can be factored into the form $(s + 2)\,(s + 3)$. Thus, $F(s)$ can be written in partial fraction form as

$$F(s) = \frac{s-3}{(s+2)(s+3)} = \frac{A}{s+2} + \frac{B}{s+3}\,. \tag{4.3}$$

where A and B are constants that must be determined.

To evaluate A, multiply both sides of (4.3) by $(s + 2)$ and then set $s = -2$. This gives

$$A = F(s)(s+2)\Big|_{s=-2} = \frac{s-3}{s+3}\Big|_{s=-2} = -5$$

and $B(s+2)/(s+3)\big|_{s=-2}$ is identically zero. In the same manner, to find the value of B we multiply both sides of (4.3) by $(s + 3)$ and get

$$B = F(s)(s+3)\Big|_{s=-3} = \frac{s-3}{s+3}\Big|_{s=-3} = 6\,.$$

The partial fraction form of (4.3) is

$$F(s) = \frac{-5}{s+2} + \frac{6}{s+3}\,.$$

The inverse transform is given by

$$f(t) = \mathscr{L}^{-1}\{F(s)\} = -5\,\mathscr{L}^{-1}\left\{\frac{1}{s+2}\right\} + 6\,\mathscr{L}^{-1}\left\{\frac{1}{s+3}\right\} = -5\,e^{-2t} + 6\,e^{-3t}$$

where entry 8 in Table 5.1, is used.

Example 5.4.2

Find the inverse Laplace transform of the function

$$F(s) = \frac{s+1}{\left[(s+2)^2+1\right](s+3)}\,.$$

Solution

This function is written in the form

$$F(s) = \frac{A}{s+3} + \frac{Bs+C}{\left[(s+2)^2+1\right]} = \frac{s+1}{\left[(s+2)^2+1\right](s+3)}\,.$$

The value of A is deduced by multiplying both sides of this equation by $(s + 3)$ and then setting $s = -3$. This gives

$$A = (s+3)F(s)\Big|_{s=-3} = \frac{-3+1}{(-3+2)^2+1} = -1\,.$$

To evaluate B and C, combine the two fractions and equate the coefficients of the powers of s in the numerators. This yields

$$\frac{-1\left[(s+2)^2+1\right] + (s+3)(Bs+C)}{\left[(s+2)^2+1\right](s+3)} = \frac{s+1}{\left[(s+2)^2+1\right](s+3)}$$

from which it follows that

$$-(s^2 + 4s + 5) + Bs^2 + (C + 3B)s + 2C = s + 1\,.$$

Combine like-powered terms to write

$$(-1 + B)s^2 + (-4 + C + 3B)s + (-5 + 3C) = s + 1 \ .$$

Therefore,

$$-1 + B = 0, \quad -4 + C + 3B = 1, \quad -5 + 3C = 1 \ .$$

From these equations we obtain

$$B = 1, \quad C = 2 \ .$$

The function $F(s)$ is written in the equivalent form

$$F(s) = \frac{-1}{s+3} + \frac{s+2}{(s+2)^2+1} \ .$$

Now using Table 5.1, the result is

$$f(t) = -e^{-3t} + e^{-2t}\cos t, \quad t > 0 \ .$$

In many cases, $F(s)$ is the quotient of two polynomials with real coefficients. If the numerator polynomials is of the same or higher degree than the denominator polynomial, first divide the numerator polynomial by the denominator polynomial; the division is carried forward until the numerator polynomial of the remainder is one degree less than the denominator. This results in a polynomial in s plus a proper fraction. The proper fraction can be expanded into a partial fraction expansion. The result of such an expansion is an expression of the form

$$F'(s) = B_0 + B_1 s + \cdots + \frac{A_1}{s-s_1} + \frac{A_2}{s-s_2} + \cdots + \frac{A_{p1}}{s-s_p} + \frac{A_{p2}}{(s-s_p)^2} + \cdots + \frac{A_{pr}}{(s-s_p)^r} \ . \qquad (4.4)$$

This expression has been written in a form to show three types of terms; polynomial, simple partial fraction including all terms with distinct roots, and partial fraction appropriate to multiple roots.

To find the constants A_1, A_2, … the polynomial terms are removed, leaving the proper fraction

$$F'(s) - (B_0 + B_1 s + \cdots) = F(s) \qquad (4.5)$$

where

$$F(s) = \frac{A_1}{s-s_1} + \frac{A_2}{s-s_2} + \cdots + \frac{A_k}{s-s_k} + \frac{A_{p1}}{s-s_p} + \frac{A_{p2}}{(s-s_p)^2} + \cdots + \frac{A_{pr}}{(s-s_p)^r} \ .$$

To find the constants A_k that are the residues of the function $F(s)$ at the simple poles s_k, it is only necessary to note that as $s \to s_k$ the term $A_k(s - s_k)$ will become large compared with all other terms. In the limit

$$A_k = \lim_{s \to s_k} (s - s_k) F(s) . \qquad (4.6)$$

Upon taking the inverse transform for each simple pole, the result will be a simple exponential of the form

$$\mathcal{L}^{-1}\left\{\frac{A_k}{s-s_k}\right\} = A_k e^{s_k t} . \tag{4.7}$$

Note also that because $F(s)$ contains only real coefficients, if s_k is a complex pole with residue A_k, there will also be a conjugate pole s_k^* with residue A_k^*. For such complex poles

$$\mathcal{L}^{-1}\left\{\frac{A_k}{s-s_k} + \frac{A_k^*}{s-s_k^*}\right\} = A_k e^{s_k t} + A_k^* e^{s_k^* t} .$$

These can be combined in the following way:

$$\begin{aligned} \text{response} &= \left(a_k + jb_k\right)e^{(\sigma_k + j\omega_k)t} + \left(a_k - jb_k\right)e^{(\sigma_k - j\omega_k)t} \\ &= e^{\sigma_k t}\left[\left(a_k + jb_k\right)\left(\cos\omega_k t + j\sin\omega_k t\right) + \left(a_k - jb_k\right)\left(\cos\omega_k t + j\sin\omega_k t\right)\right] \\ &= 2e^{\sigma_k t}\left(a_k \cos\omega_k t - b_k \sin\omega_k t\right) \\ &= 2A_k e^{\sigma_k t}\cos\left(\omega_k t + \theta_k\right) \end{aligned} \tag{4.8}$$

where $\theta_k = \tan^{-1}(b_k/a_k)$ and $A_k = a_k/\cos\theta_k$.

When the proper fraction contains a multiple pole of order r, the coefficients in the partial-fraction expansion $A_{p1}, A_{p2}, \ldots, A_{pr}$ that are involved in the terms

$$\frac{A_{p1}}{\left(s-s_p\right)} + \frac{A_{p2}}{\left(s-s_p\right)^2} + \cdots + \frac{A_{pr}}{\left(s-s_p\right)^r}$$

must be evaluated. A simple application of (4.6) is not adequate. Now the procedure is to multiply both sides of (4.5) by $(s - s_p)^r$, which gives

$$\begin{aligned} \left(s-s_p\right)^r F(s) &= \left(s-s_p\right)^r \left[\frac{A_1}{s-s_1} + \frac{A_2}{s-s_2} + \cdots + \frac{A_k}{s-s_k}\right] + A_{p1}\left(s-s_p\right)^{r-1} + \cdots \\ &\quad + A_{p(r-1)}\left(s-s_p\right) + A_{pr} \end{aligned} \tag{4.9}$$

In the limit as $s = s_p$ all terms on the right vanish with the exception of A_{pr}. Suppose now that this equation is differentiated once with respect to s. The constant A_{pr} will vanish in the differentiation but $A_{p(r-1)}$ will be determined by setting $s = s_p$. This procedure will be continued to find each of the coefficients A_{pk}. Specifically, the procedure is specified by

$$A_{pk} = \frac{1}{(r-k)!}\left\{\frac{d^{r-k}}{ds^{r-k}} F(s)\left(s-s_p\right)^r\right\}_{s=s_p} , \quad k = 1, 2, \ldots, r . \tag{4.10}$$

Example 5.4.3

Find the inverse transform of the following function:

$$F(s)=\frac{s^3+2s^2+3s+1}{s^2(s+1)}.$$

Solution

This is not a proper fraction. The numerator polynomial is divided by the denominator polynomial by simple long division. The result is

$$F(s)=1+\frac{s^2+3s+1}{s^2(s+1)}.$$

The proper fraction is expanded into partial fraction form

$$F_p(s)=\frac{s^2+3s+1}{s^2(s+1)}=\frac{A_{11}}{s}+\frac{A_{12}}{s^2}+\frac{A_2}{s+1}.$$

The value of A_2 is deduced using (4.6)

$$A_2=\left[(s+1)F_p(s)\right]_{s=-1}=\left.\frac{s^2+3s+1}{s^2}\right|_{s=-1}=-1.$$

To find A_{11} and A_{12} we proceed as specified in (4.10)

$$A_{12}=\left[s^2F_p(s)\right]_{s=0}=\left.\frac{s^2+3s+1}{s+1}\right|_{s=0}=1$$

$$A_{11}=\frac{1}{1!}\left\{\frac{d}{ds}s^2F_p(s)\right\}_{s=0}=\frac{d}{ds}\left[\frac{s^2+3s+1}{s+1}\right]_{s=0}=\left.-\frac{s^2+3s+1}{(s+1)^2}+\frac{2s+3}{s+1}\right|_{s=0}=4.$$

Therefore,

$$F(s)=1+\frac{4}{s}+\frac{1}{s^2}-\frac{1}{s+1}.$$

From Table 5.1 the inverse transform is

$$f(t)=\delta(t)+4+t-e^{-t}, \quad \text{for } t\geq 0.$$

If the function $F(s)$ exists in proper fractional form as the quotient of two polynomials, we can employ the Heaviside expansion theorem in the determination of $f(t)$ from $F(s)$. This theorem is an efficient method for finding the residues of $F(s)$. Let

$$F(s)=\frac{P(s)}{Q(s)}=\frac{A_1}{s-s_1}+\frac{A_2}{s-s_2}+\cdots+\frac{A_k}{s-s_k}$$

where $P(s)$ and $Q(s)$ are polynomials with no common factors and with the degree of $P(s)$ less than the degree of $Q(s)$.

Suppose that the factors of $Q(s)$ are distinct constants. Then, as in (4.6) we find

$$A_k = \lim_{s \to s_k} \left[\frac{s - s_k}{Q(s)} P(s) \right].$$

Also, the limit $P(s)$ is $P(s_k)$. Now, because

$$\lim_{s \to s_k} \frac{s - s_k}{Q(s)} = \lim_{s \to s_k} \frac{1}{Q^{(1)}(s)} = \frac{1}{Q^{(1)}(s_k)},$$

then

$$A_k = \frac{P(s_k)}{Q^{(1)}(s_k)}.$$

Thus,

$$F(s) = \frac{P(s)}{Q(s)} = \sum_{n=1}^{k} \frac{P(s_n)}{Q^{(1)}(s_n)} \cdot \frac{1}{(s - s_n)}. \tag{4.11}$$

From this, the inverse transformation becomes

$$f(t) = \mathscr{L}^{-1}\left\{ \frac{P(s)}{Q(s)} \right\} = \sum_{n=1}^{k} \frac{P(s_n)}{Q^{(1)}(s_n)} e^{s_n t}.$$

This is the Heaviside expansion theorem. It can be written in formal form.

Theorem 5.4.1 Heaviside Expansion Theorem

If $F(s)$ is the quotient $P(s)/Q(s)$ of two polynomials in s such that $Q(s)$ has the higher degree and contains simple poles the factor $s - s_k$, which are not repeated, then the term in $f(t)$ corresponding to this factor can be written $\dfrac{P(s_k)}{Q^{(1)}(s_k)} e^{s_k t}$.

Example 5.4.4

Repeat Example 4.1 employing the Heaviside expansion theorem.

Solution

We write (4.2) in the form

$$F(s) = \frac{P(s)}{Q(s)} = \frac{s-3}{s^2 + 5s + 6} = \frac{s-3}{(s+2)(s+3)}.$$

The derivative of the denominator is

$$Q^{(1)}(s) = 2s + 5$$

from which, for the roots of this equation,

$$Q^{(1)}(-2) = 1, \quad Q^{(1)}(-3) = -1 \ .$$

Hence,

$$P(-2) = -5, \quad P(-3) = -6 \ .$$

The final value for $f(t)$ is

$$f(t) = -5e^{-2t} + 6\,e^{-3t} .$$

Example 5.4.5

Find the inverse Laplace transform of the following function using the Heaviside expansion theorem:

$$\mathscr{L}^{-1}\left\{\frac{2s+3}{s^2+4s+7}\right\}.$$

Solution

The roots of the denominator are

$$s^2+4s+7=\left(s+2+j\sqrt{3}\right)\left(s+2-j\sqrt{3}\right).$$

That is, the roots of the denominator are complex. The derivative of the denominator is

$$Q^{(1)}(s) = 2s + 4 \ .$$

We deduce the values $P(s)/Q^{(1)}(s)$ for each root

$$\begin{array}{lll} \text{For } s_1=-2-j\sqrt{3} & Q^{(1)}\left(s_1\right)=-j2\sqrt{3} & P\left(s_1\right)=-1-j2\sqrt{3} \\ \text{For } s_2=-2+j/3 & Q^{(1)}\left(s_2\right)=+j2\sqrt{3} & P\left(s_2\right)=-1+j2\sqrt{3} \ . \end{array}$$

Then

$$\begin{aligned} f(t) &= \frac{-1-j2\sqrt{3}}{-j2\sqrt{3}}e^{\left(-2-j2\sqrt{3}\right)t}+\frac{-1+j2\sqrt{3}}{j2\sqrt{3}}e^{\left(-2+j2\sqrt{3}\right)t} \\ &= e^{-2t}\left[\frac{-1-j2\sqrt{3}}{-j2\sqrt{3}}e^{-j2\sqrt{3}t}+\frac{-1+j2\sqrt{3}}{j2\sqrt{3}}e^{j2\sqrt{3}t}\right] \\ &= e^{-2t}\left[\frac{\left(e^{-j2\sqrt{3}t}-e^{j2\sqrt{3}t}\right)}{j2\sqrt{3}}+\left(e^{-j2\sqrt{3}t}+e^{j2\sqrt{3}t}\right)\right] \\ &= e^{-2t}\left(2\cos 2\sqrt{3t}-\frac{1}{\sqrt{3}}\sin 2\sqrt{3t}\right) \end{aligned}$$

5.5 Solution of Ordinary Linear Equations with Constant Coefficients

The Laplace transform is used to solve homogeneous and nonhomogeneous ordinary different equations or systems of such equations. To understand the procedure, we consider a number of examples.

Example 5.5.1

Find the solution to the following differential equation subject to prescribed initial conditions: $y(0+)$; $(dy/dt) + ay = x(t)$.

Solution

Laplace transform this differential equation. This is accomplished by multiplying each term by $e^{-st}dt$ and integrating form 0 to ∞. The result of this operation is

$$sY(s) - y(0+) + aY(s) = X(s)\,,$$

from which

$$Y(s) = \frac{X(s)}{s+a} + \frac{y(0+)}{s+a}\,.$$

If the input $x(t)$ is the unit step function $u(t)$, then $X(s) = 1/s$ and the final expression for $Y(s)$ is

$$Y(s) = \frac{1}{s(s+a)} + \frac{y(0+)}{s+a}\,.$$

Upon taking the inverse transform of this expression

$$y(t) = \mathscr{L}^{-1}\{Y(s)\} = \mathscr{L}^{-1}\left(\frac{1}{a}\left[\frac{1}{s} - \frac{1}{s+a}\right] + \frac{y(0+)}{s+a}\right)$$

with the result

$$y(t) = \frac{1}{a}\left(1 - e^{-at}\right) + y(0+)e^{-at}\,.$$

Example 5.5.2

Find the general solution to the differential equation

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 4y = 10$$

subject to zero initial conditions.

Solution

Laplace transform this differential equation. The result is

$$s^2Y(s) + 5sY(s) + 5Y(s) = \frac{10}{s}\,.$$

Solving for $Y(s)$, we get

$$Y(s)=\frac{10}{s(s^2+5s+4)}=\frac{10}{s(s+1)(s+4)}.$$

Expand this into partial-fraction form, thus

$$Y(s)=\frac{A}{s+1}+\frac{B}{s+4}+\frac{C}{s}.$$

Then

$$A=Y(s)(s+1)\Big|_{s=-1}=\frac{10}{s(s+4)}\Bigg|_{s=-1}=-\frac{10}{3}$$

$$B=Y(s)(s+4)\Big|_{s=-4}=\frac{10}{s(s+1)}\Bigg|_{s=-4}=\frac{10}{12}$$

$$C=sY(s)\Big|_{s=0}=\frac{10}{(s+1)(s+4)}\Bigg|_{s=0}=\frac{10}{4}$$

and

$$Y(s)=10\left[-\frac{1}{3(s+1)}+\frac{1}{12(s+4)}+\frac{1}{4s}\right].$$

The inverse transform is

$$x(t)=10\left[-\frac{1}{3}e^{-t}+\frac{1}{12}e^{-4t}+\frac{1}{4}\right].$$

Example 5.5.3

Find the velocity of the system shown in Figure 5.6a when the applied force is $f(t) = e^{-t}u(t)$. Assume zero initial conditions. Solve the same problem using convolution techniques. The input is the force and the output is the velocity.

Solution

The controlling equation is, from Figure 5.6b,

$$\frac{dv}{dt}+5v+4\int_0^t v\,dt=e^{-t}u(t).$$

Laplace transform this equation and then solve for $F(s)$. We obtain

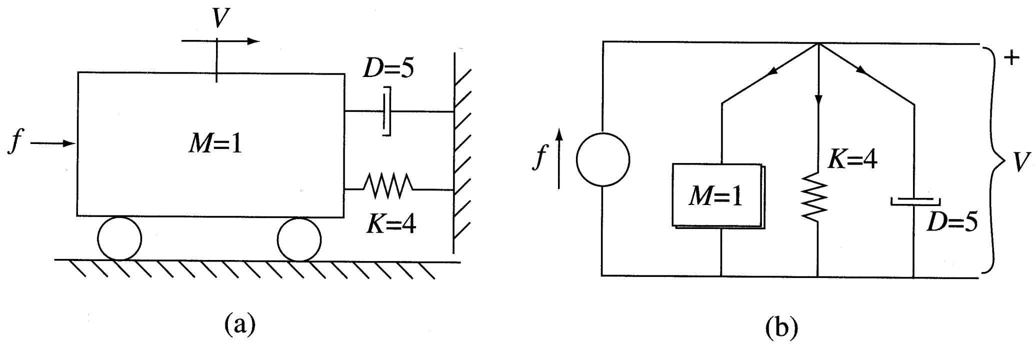


FIGURE 5.6 The mechanical system and its network equivalent.

$$V(s)=\frac{s}{(s+1)(s^2+5s+4)}=\frac{s}{(s+1)^2(s+4)}.$$

Write this expression in the form

$$V(s)=\frac{A}{s+4}+\frac{B}{s+1}+\frac{C}{(s+1)^2}$$

where

$$A=\left.\frac{s}{(s+1)^2}\right|_{s=-4}=-\frac{4}{9}$$

$$B=\frac{1}{1!}\frac{d}{ds}\left.\left(\frac{s}{s+4}\right)\right|_{s=-1}=\frac{4}{9}$$

$$C=\left.\frac{s}{s+4}\right|_{s=-1}=-\frac{1}{3}.$$

The inverse transform of $V(s)$ is given by

$$v(t)=-\frac{4}{9}e^{-4t}+\frac{4}{9}e^{-t}-\frac{1}{3}te^{-t},\quad t\geq 0.$$

To find $v(t)$ by the use of the convolution integral, we first find $h(t)$, the impulse response of the system. The quantity $h(t)$ is specified by

$$\frac{dh}{dt}+5h+4\int h\,dt=\delta(t)$$

where the system is assumed to be initially relaxed. The Laplace transform of this equation yields

$$H(s)=\frac{s}{s^2+5s+4}=\frac{s}{(s+4)(s+1)}=\frac{4}{3}\frac{1}{s+4}-\frac{1}{3}\frac{1}{s+1}.$$

The inverse transform of this expression is easily found to be

$$h(t)=\frac{4}{3}e^{-4t}-\frac{1}{3}e^{-t}, \quad t\geq 0.$$

The output of the system to the input $e^{-t}u(t)$ is written

$$v(t)=\int_{-\infty}^{\infty}h(\tau)f(t-\tau)d\tau=\int_0^t e^{-(t-\tau)}\left[\frac{4}{3}e^{-4\tau}-\frac{1}{3}e^{-\tau}\right]d\tau$$

$$=e^{-t}\left[\frac{4}{3}\int_0^t e^{-3\tau}d\tau-\frac{1}{3}\int_0^t d\tau\right]=e^{-t}\left[\frac{4}{3}\left(\frac{1}{-3}\right)e^{-3\tau}\Big|_0^t-\frac{1}{3}t\right]$$

$$=-\frac{4}{9}e^{-4t}+\frac{4}{9}e^{-t}-\frac{1}{3}te^{-t}, \quad t\geq 0.$$

This result is identical with that found using the Laplace transform technique.

Example 5.5.4

Find an expression for the voltage $v_2(t)$ for $t>0$ in the circuit of Figure 5.7. The source $v_1(t)$, the current $i_L(0-)$ through $L = 2H$, and the voltage $v_c(0-)$ across the capacitor $C = 1\ F$ at the switching instant are all assumed to be known.

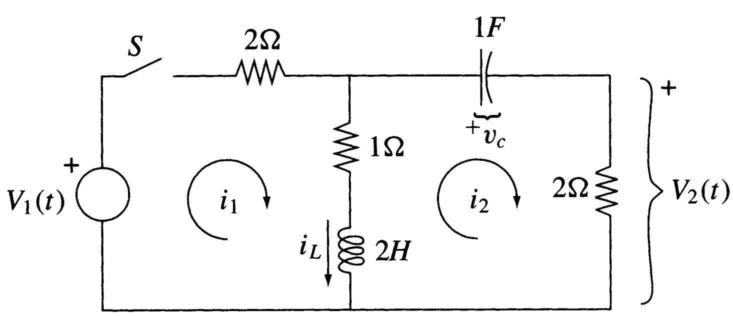


FIGURE 5.7 The circuit for Example 5.5.4.

Solution

After the switch is closed, the circuit is described by the loop equations

$$\left(3i_1+\frac{2di_1}{dt}\right)-\left(1i_2+\frac{2di_2}{dt}\right)=v_2(t)$$

$$-\left(1i_1+\frac{2di_1}{dt}\right)+\left(3i_2+\frac{2di_2}{dt}+\int i_2dt\right)=0$$

$$v_2(t)=2i_2(t).$$

All terms in these equations are Laplace transformed. The result is the set of equations

$$(3+2s)I_1(s)-(1+2s)I_2(s)=V_1(s)+2\left[i_1(0+)-i_2(0+)\right]$$

$$-(1+2s)I_1(s)+\left(3+2s+\frac{1}{s}\right)I_2(s)=2\left[-i_1(0+)+i_2(0+)\right]-\frac{q_2(0+)}{s}$$

$$V_2(s)=2I_2(s).$$

The current through the inductor is

$$i_L(t) = i_1(t) - i_2(t)\,.$$

At the instant $t = 0+$

$$i_L(0+) = i_1(0+) - i_2(0+)\,.$$

Also, because

$$\begin{aligned}\frac{1}{C}q_2(t) &= \frac{1}{C}\int_{-\infty}^{t} i_2(t)\,dt \\ &= \frac{1}{C}\lim_{t\to 0+}\int_{0}^{t} i_2(t)\,dt+\frac{1}{C}\int_{-\infty}^{0} i_2(t)\,dt = 0+v_c(0-),\end{aligned}$$

then

$$\frac{q_2(0+)}{C}\triangleq v_c(0+)=v_c(0-)=i_2^{(-1)}(0)=\frac{q_2(0+)}{1}.$$

The equation set is solved for $I_2(s)$, which is written by Cramer's rule

$$I_2(s)=\frac{\begin{vmatrix}3+2s & V_1(s)+2i_L(0+)\\ -(1+2s) & -2i_L(0+)-\dfrac{v_c(0+)}{s}\end{vmatrix}}{\begin{vmatrix}3+2s & -(1+2s)\\ -(1+2s) & 3+2s+\dfrac{1}{s}\end{vmatrix}}$$

$$=\frac{(3+2s)\left[-2i_L(0+)-\dfrac{v_c(0+)}{s}\right]+(1+2s)\left[V_1(s)+2i_L(0+)\right]}{(3+2s)\left(\dfrac{2s^2+3s+1}{s}\right)-(1+2s)^2}$$

$$=\frac{-(2s^2+3s)v_c(0+)-4si_L(0+)+(2s^2+s)V_1(s)}{8s^2+10s+3}.$$

Further

$$V_2(s) = 2I_2(s).$$

Then, upon taking the inverse transform

$$v_1(t) = 2\mathcal{L}^{-1}\{I_2(s)\}.$$

If the circuit contains no stored energy at $t = 0$, then $i_L(0+) = v_c(0+) = 0$ and now

$$v_2(t) = 2\mathcal{L}^{-1}\left\{\frac{(2s^2+s)V_1(s)}{8s^2+10s+3}\right\}.$$

For the particular case when $v_1 = u(t)$ so that $V_1(s) = 1/s$

$$v_2(t) = 2\mathcal{L}^{-1}\left\{\frac{2s+1}{8s^2+10s+3}\right\} = 2\mathcal{L}^{-1}\left\{\frac{2s+1}{8\left(s+\frac{1}{2}\right)(s+3/4)}\right\}$$

$$= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+\frac{3}{4}}\right\} = \frac{1}{2}e^{-3t/4}, \quad t \geq 0.$$

The validity of this result is readily confirmed because at the instant $t = 0+$ the inductor behaves as an open circuit and the capacitor behaves as a short circuit. Thus, at this instant, the circuit appears as two equal resistors in a simple series circuit and the voltage is shared equally.

Example 5.5.5

The input to the RL circuit shown in Figure 5.8a is the recurrent series of impulse functions shown in Figure 5.8b. Find the output current.

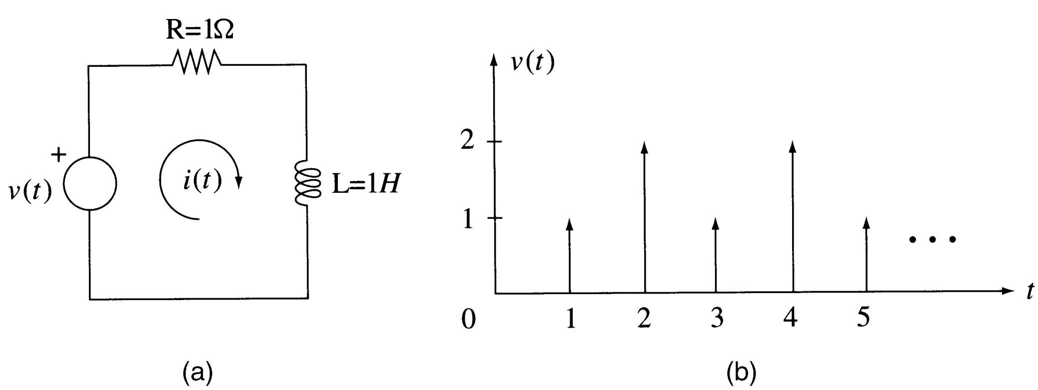


FIGURE 5.8 (a) The circuit, (b) the input pulse train.

Solution

The differential equation that characterizes the system is

$$\frac{di(t)}{dt} + i(t) = v(t).$$

For zero initial current through the inductor, the Laplace transform of the equation is

$$(s + 1)I(s) = V(s).$$

Now, from the fact that $\mathscr{L}\{\delta(t)\} = 1$ and the shifting property of Laplace transforms, we can write the explicit form for $V(s)$, which is

$$\begin{aligned} V(s) &= 2 + e^{-s} + 2e^{-2s} + e^{-3s} + 2e^{-4s} + \cdots \\ &= \left(2 + e^{-s}\right)\left(1 + e^{-2s} + e^{-4s} + \cdots\right) \\ &= \frac{2 + e^{-s}}{1 - e^{-2s}}. \end{aligned}$$

Thus, we must evaluate $i(t)$ from

$$I(s) = \frac{2 + e^{-s}}{1 - e^{-2s}}\frac{1}{s+1} = \frac{2}{\left(1 - e^{-2s}\right)(s+1)} + \frac{e^{-s}}{\left(1 - e^{-2s}\right)(s+1)}.$$

Expand these expressions into

$$I(s) = \frac{2}{s+1}\left(1 + e^{-2s} + e^{-4s} + e^{-6s} + \cdots\right) + \frac{1}{s+1}\left(e^{-s} + e^{-3s} + e^{-5s} + e^{-7s} + \cdots\right).$$

The inverse transform of these expressions yields

$$\begin{aligned} i(t) &= 2e^{-t}u(t) + 2e^{-(t-2)}u(t-2) + 2e^{-(t-4)}u(t-4) + \cdots \\ &\quad + e^{-(t-1)}u(t-1) + e^{-(t-3)}u(t-3) + e^{-(t-5)}u(t-5) + \cdots \end{aligned}$$

The result has been sketched in Figure 5.9.

5.6 The Inversion Integral

The discussion in Section 5.3 related the inverse Laplace transform to the direct Laplace transform by the expressions

$$F(s) = \mathscr{L}\{f(t)\} \tag{6.1a}$$

$$f(t) = \mathscr{L}^{-1}\{F(s)\}. \tag{6.1b}$$

The subsequent discussion indicated that the use of equation (6.1b) suggested that the $f(t)$ so deduced was unique; that there was no other $f(t)$ that yielded the specified $F(s)$. We found that although $f(t)$ represents a real function of the positive real variable t, the transform $F(s)$ can assume a complex variable form. What this means, of course, is that a mathematical form for the inverse Laplace transform was not essential for linear functions that satisfied the Dirichlet conditions. In some cases, Table 5.1 is not adequate

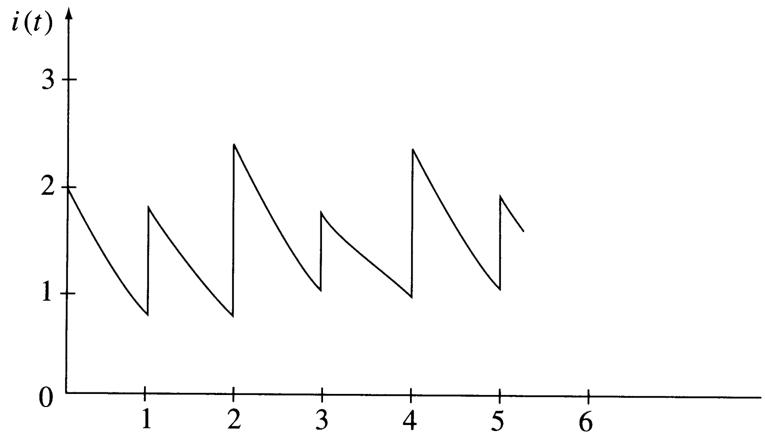


FIGURE 5.9 The response of the RL circuit to the pulse train.

for many functions when s is a complex variable and an analytic form for the inversion process of (6.1b) is required.

To deduce the complex inversion integral, we begin with the Cauchy second integral theorem, which is written

$$\oint \frac{F(z)}{s-z}\,dz = j2\pi F(s)$$

where the contour encloses the singularity at s. The function $F(s)$ is analytic in the half-plane $\text{Re}(s) \geq c$. If we apply the inverse Laplace transformation to the function s on both sides of this equation, we can write

$$j2\pi\,\mathscr{L}^{-1}\{F(s)\} = \lim_{\omega\to\infty}\int_{\sigma-j\omega}^{\sigma+j\omega} F(z)\,\mathscr{L}^{-1}\left\{\frac{1}{s-z}\right\} dz.$$

But $F(s)$ is the Laplace transform of $f(t)$; also, the inverse transform of $1/(s-z)$ is e^{zt}. Then it follows that

$$f(t) = \frac{1}{2\pi j}\lim_{\omega\to\infty}\int_{\sigma-j\omega}^{\sigma+j\omega} e^{zt}F(z)\,dz = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} e^{zt}F(z)\,dz. \tag{6.2}$$

This equation applies equally well to both the one-sided and the two-sided transforms.

It was pointed out in Section 5.1 that the path of integration (6.2) is restricted to value of σ for which the direct transform formula converges. In fact, for the two-sided Laplace transform, the region of convergence must be specified in order to determine uniquely the inverse transform. That is, for the two-sided transform, the regions of convergence for functions of time that are zero for $t > 0$, zero for $t < 0$, or in neither category, must be distinguished. For the one-sided transform, the region of convergence is given by σ, where σ is the abscissa of absolute convergence.

The path of integration in (6.2) is usually taken as shown in Figure 5.10 and consists of the straight line ABC displayed to the right of the origin by σ and extending in the limit from $-j\infty$ to $+j\infty$ with connecting semicircles. The evaluation of the integral usually proceeds by using the Cauchy integral theorem, which specifies that

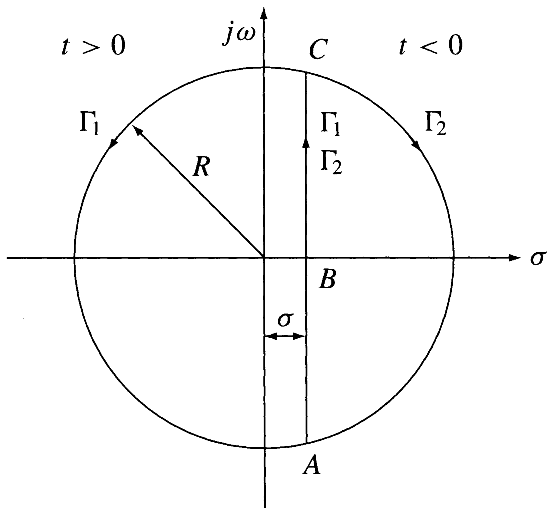


FIGURE 5.10 The path of integration in the *s*-plane.

$$f(t)=\frac{1}{2\pi j}\lim_{R\to\infty}\oint_{\Gamma_1}F(s)e^{st}ds$$
$$=\sum \text{residues of } F(s)e^{st} \text{ at the singularities to the left of } ABC; \quad t>0. \tag{6.3}$$

But the contribution to the integral around the circular path with $R \to \infty$ is zero, leaving the desired integral along the path *ABC*, and

$$f(t)=\frac{1}{2\pi j}\lim_{R\to\infty}\oint_{\Gamma_2}F(s)e^{st}ds$$
$$=-\sum \text{residues of } F(s)e^{st} \text{ at the singularities to the right of } ABC; \quad t<0. \tag{6.4}$$

We will present a number of examples involving these equations.

Example 5.6.1

Use the inversion integral to find $f(t)$ for the function

$$F(s)=\frac{1}{s^2+w^2}.$$

Note that by entry 15 of Table 5.1, this is sin wt/w.

Solution

The inversion integral is written in a form that shows the poles of the integrand.

$$f(t)=\frac{1}{2\pi j}\oint\frac{e^{st}}{(s+jw)(s-jw)}ds.$$

The path chosen is Γ_1 in Figure 5.10. Evaluate the residues

$$\operatorname{Res}\left[(s-jw)\frac{e^{st}}{s^2+w^2}\right]_{s=jw}=\left.\frac{e^{st}}{s+jw}\right|_{s=jw}=\frac{e^{jwt}}{2wj}$$

$$\operatorname{Res}\left[(s+jw)\frac{e^{st}}{s^2+w^2}\right]_{s=-jw}=\left.\frac{e^{st}}{s-jw}\right|_{s=-jw}=\frac{e^{-jwt}}{-2wj}.$$

Therefore,

$$f(t)=\sum\operatorname{Res}=\frac{e^{jwt}-e^{-jwt}}{2jw}=\frac{\sin wt}{w}.$$

Example 5.6.2

Evaluate $\mathcal{L}^{-1}\{1/\sqrt{s}\}$.

Solution

The function $F(s) = 1/\sqrt{s}$ is a double-valued function because of the square root operation. That is, if s is represented in polar form by $re^{j\theta}$, then $re^{j(\theta+2\pi)}$ is a second acceptable representation, and $\sqrt{s}=\sqrt{re^{j(\theta+2\pi)}}=-\sqrt{re^{j\theta}}$, thus showing two different values for $\sqrt{s}$. But a double-valued function is not analytic and requires a special procedure in its solution.

The procedure is to make the function analytic by restricting the angle of s to the range $-\pi < \theta < \pi$ and by excluding the point $s = 0$. This is done by constructing a branch cut along the negative real axis, as shown in Figure 5.11. The end of the branch cut, which is the origin in this case, is called a branch point. Because a branch cut can never be crossed, this essentially ensures that $F(s)$ is single valued. Now, however, the inversion integral (6.3) becomes for $t > 0$

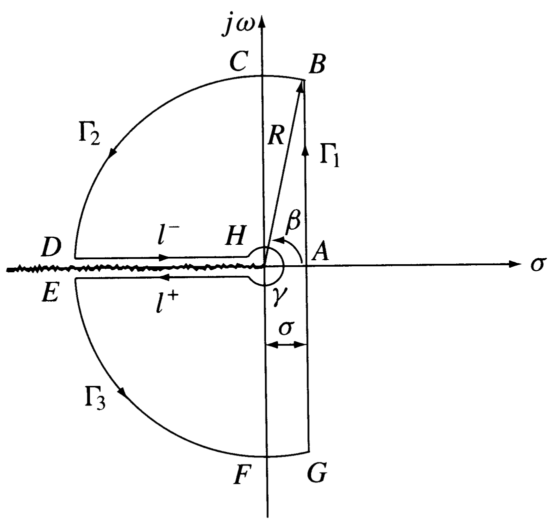


FIGURE 5.11 The integration contour for $\mathcal{L}^{-1}\{1/\sqrt{s}\}$.

$$f(t) = \lim_{R\to\infty} \frac{1}{2\pi j}\int_{GAB} F(s)e^{st}ds = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} F(s)e^{st}ds$$
$$= -\frac{1}{2\pi j}\left[\int_{BC} + \int_{\Gamma_2} + \int_{\ell-} + \int_{\gamma} + \int_{\ell+} + \int_{\Gamma_3} + \int_{FG}\right], \tag{6.5}$$

which does not include any singularity.

First we will show that for $t > 0$ the integrals over the contours BC and CD vanish as $R \to \infty$, from which $\int_{\Gamma_2} = \int_{\Gamma_3} = \Gamma_{BC} = \int_{FG} = 0$. Note from Figure 5.11 that $\beta = \cos^{-1}(\sigma/R)$ so that the integral over the arc BC is, because $|e^{j\theta}| = 1$,

$$|I| \le \int_{BC} \left| \frac{e^{\sigma t} e^{j\omega t}}{R^{\frac{1}{2}} e^{j\theta/2}} jRe^{j\theta}\, d\theta \right| = e^{\sigma t}\sqrt{R}\int_{\beta}^{\pi\ 2} d\theta = e^{\sigma t}\sqrt{R}\left(\frac{\pi}{2} - \cos^{-1}\frac{\sigma}{R}\right)$$
$$= e^{\sigma t}\sqrt{R}\sin^{-1}\frac{\sigma}{R}$$

But for small arguments $\sin^{-1}(\sigma/R) = \sigma/R$, and in the limit as $R \to \infty$, $I \to 0$. By a similar approach, we find that the integral over CD is zero. Thus, the integrals over the contours Γ_2 and Γ_3 are also zero as $R \to \infty$.

For evaluating the integral over γ, let $s = re^{j\theta} = r(\cos\theta + j\sin\theta)$ and

$$\int_{\gamma} F(s)e^{st}ds = \int_{-\pi}^{\pi} \frac{e^{r(\cos\theta + j\sin\theta)t}}{\sqrt{r e^{j\theta/2}}} jre^{j\theta}\, d\theta = 0 \quad \text{as } r \to 0.$$

The remaining integrals in (6.5) are written

$$f(t) = -\frac{1}{2\pi j}\left[\int_{\ell-} F(s)e^{st}ds + \int_{\ell+} F(s)e^{st}ds\right]. \tag{6.6}$$

Along path $l-$, let $s = ue^{j\pi} = -u$; $\sqrt{s} = j\sqrt{u}$, and $ds = -du$, where u and $\sqrt{u}$ are real positive quantities. Then

$$\int_{\ell-} F(s)e^{st}ds = -\int_{\infty}^{0} \frac{e^{-ut}}{j\sqrt{u}}du = \frac{1}{j}\int_{0}^{\infty} \frac{e^{-ut}}{j\sqrt{u}}du.$$

Along path $l+$, $s = -ue^{j2\pi} = -u$, $\sqrt{s} = -j\sqrt{u}$ (not $+j\sqrt{u}$), and $ds = -du$. Then

$$\int_{\ell+} F(s)e^{st}ds = -\int_{0}^{\infty} \frac{e^{-ut}}{-j\sqrt{u}}du = \frac{1}{j}\int_{0}^{\infty} \frac{e^{-ut}}{j\sqrt{u}}du.$$

Combine these results to find

$$f(t) = -\frac{1}{2\pi j}\left[\frac{2}{j}\int_0^{\infty} u^{-\frac{1}{2}} e^{-ut} du\right] = \frac{1}{\pi}\int_0^{\infty} u^{-\frac{1}{2}} e^{-ut} du,$$

which is a standard form integral with the value

$$f(t) = \frac{1}{\pi}\sqrt{\frac{\pi}{t}} = \frac{1}{\sqrt{\pi t}}, \quad t > 0.$$

Example 5.6.3

Find the inverse Laplace transform of the function

$$F(s) = \frac{1}{s\left(1+e^{-s}\right)}.$$

Solution

The integrand in the inversion integral $\frac{e^{st}}{s\left(1+e^{-s}\right)}$ possesses simple poles at: $s = 0$ and $s = jn\pi$, $n = \pm 1, \pm 3, \pm \cdots$ (odd values). These are illustrated in Figure 5.12. We see that the function $e^{st}/s(1 + e^{-s})$ is analytic in the s-plane except at the simple poles at $s = 0$ and $s = jn\pi$. Hence, the integral is specified in terms of the residues in the various poles. We have, specifically

$$\text{Res}\left\{\frac{se^{st}}{s\left(1+e^{-s}\right)}\right\}\Bigg|_{s=0} = \frac{1}{2} \quad \text{for } s = 0$$

$$\text{Res}\left\{\frac{(s-jn)e^{st}}{s\left(1+e^{-s}\right)}\right\}\Bigg|_{s=jn} = \frac{0}{0} \quad \text{for } s = jn\pi. \tag{6.7}$$

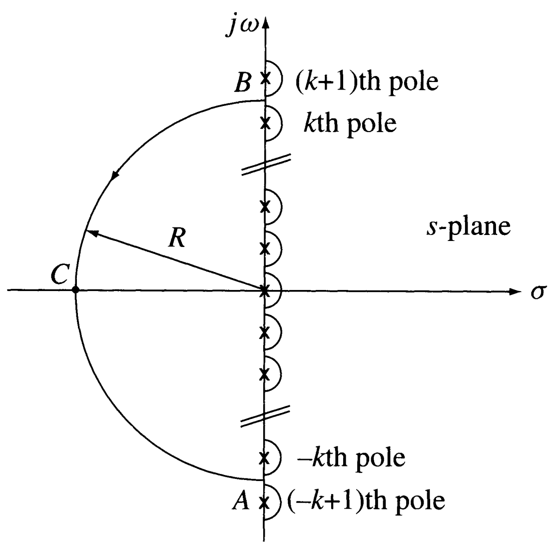


FIGURE 5.12 The pole distribution of the given function.

The problem we now face in this evaluation is that

$$\operatorname{Res}\left\{(s-a)\frac{n(s)}{d(s)}\right\}\Bigg|_{s=a}=\frac{0}{0}$$

where the roots of $d(s)$ are such that $s = a$ cannot be factored. However, we know from complex function theory that

$$\frac{d[d(s)]}{ds}\Bigg|_{s=a}=\lim_{s\to a}\frac{d(s)-d(a)}{s-a}=\lim_{s\to a}\frac{d(s)}{s-a}$$

because $d(a) = 0$. Combine this result with the above equation to obtain

$$\operatorname{Res}\left\{(s-a)\frac{n(s)}{d(s)}\right\}\Bigg|_{s=a}=\frac{n(s)}{\frac{d}{ds}[d(s)]}\Bigg|_{s=a}. \tag{6.8}$$

By combining (6.8) with (6.7), we obtain

$$\operatorname{Res}\left\{\frac{e^{st}}{s\frac{d}{ds}\left(1+e^{-s}\right)}\right\}\Bigg|_{s=jn\pi}=\frac{e^{jn\pi t}}{jn\pi}\quad n\text{ odd}.$$

We obtain, by adding all of the residues,

$$f(t)=\frac{1}{2}+\sum_{n=-\infty}^{\infty}\frac{e^{jn\pi t}}{jn\pi}.$$

This can be rewritten as follows

$$\begin{aligned}f(t)&=\frac{1}{2}+\left[\cdots+\frac{e^{-j3\pi t}}{-j3\pi}+\frac{e^{-j\pi t}}{-j\pi}+\frac{e^{j\pi t}}{j\pi}+\frac{e^{j3\pi t}}{j3\pi}+\cdots\right]\\&=\frac{1}{2}+\sum_{\substack{n=1\\ n\text{ odd}}}^{\infty}\frac{2j\sin n\pi t}{jn\pi}.\end{aligned}$$

This assumes the form

$$f(t)=\frac{1}{2}+\frac{2}{\pi}\sum_{k=1}^{\infty}\frac{\sin(2k-1)\pi t}{2k-1}. \tag{6.9}$$

As a second approach to a solution to this problem, we will show the details in carrying out the contour integration for this problem. We choose the path shown in Figure 5.12 that includes semicircular hooks around each pole, the vertical connecting line from hook to hook, and the semicircular path at $R \to \infty$. Thus, we examine

$$f(t) = \frac{1}{2\pi j}\oint \frac{e^{st}}{s(1+e^{-s})}ds$$
$$= \frac{1}{2\pi j}\left[\underbrace{\int_{BCA}}_{I_1} + \underbrace{\int_{\text{vertical connecting lines}}}_{I_2} + \sum \underbrace{\int_{\text{Hooks}}}_{I_3} - \sum \text{Res}\right]. \tag{6.10}$$

We consider the several integrals in this equation.

Integral I_1. By setting $s = re^{j\theta}$ and taking into consideration that cos θ = –cos θ for $\theta > \pi/2$, the integral $I_1 \to 0$ as $r \to \infty$.

Integral I_2. Along the Y-axis, $s = jy$ and

$$I_2 = j\int_{\substack{-\infty\\ r\to 0}}^{\infty} \frac{e^{jyt}}{jy(1+e^{-jy})}dy.$$

Note that the integrand is an odd function, whence $I_2 = 0$.

Integral I_3. Consider a typical hook at $s = jn\pi$. The result is

$$\lim_{\substack{r\to 0\\ s\to jn\pi}}\left[\frac{(s-jn)e^{st}}{s(1+e^{-s})}\right] = \frac{0}{0}.$$

This expression is evaluated (as for (6.7)) and yields $e^{jn\pi t}/jn\pi$. Thus, for all poles

$$I_3 = \frac{1}{2\pi j}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{e^{st}}{s(1+e^{-s})}ds = \frac{j\pi}{2\pi j}\left[\sum_{\substack{n=-\infty\\ n\text{ odd}}}^{\infty} \frac{e^{jn\pi t}}{jn\pi} + \frac{1}{2}\right] = \frac{1}{2}\left[\frac{1}{2} + \frac{2}{\pi}\sum_{\substack{n=1\\ n\text{ odd}}}^{\infty} \frac{\sin n\pi t}{n}\right].$$

Finally, the residues enclosed within the contour are

$$\text{Res}\frac{e^{st}}{s(1+e^{-s})} = \frac{1}{2} + \sum_{\substack{n=-\infty\\ n\text{ odd}}}^{\infty} \frac{e^{jn\pi t}}{jn\pi} = \frac{1}{2} + \frac{2}{\pi}\sum_{\substack{n=1\\ n\text{ odd}}}^{\infty} \frac{\sin n\pi t}{n},$$

which is seen to be twice the value around the hooks. Then when all terms are included in (6.10), the final result is

$$f(t) = \frac{1}{2} + \frac{2}{\pi}\sum_{\substack{n=1\\ n\text{ odd}}}^{\infty} \frac{\sin n\pi t}{n} = \frac{1}{2} + \frac{2}{\pi}\sum_{k=1}^{\infty} \frac{\sin(2k-1)\pi t}{2k-1}.$$

We now shall show that the direct and inverse transforms specified by (4.1) and listed in Table 5.1 constitute unique pairs. In this connection, we see that (6.2) can be considered as proof of the following theorem:

Theorem 5.6.1

Let $F(s)$ be a function of a complex variable s that is analytic and of order $O(s^{-k})$ in the half-plane Re(s) $\geq c$, where c and k are real constants, with $k > 1$. The inversion integral (6.2) written $\mathscr{L}_t^{-1}\{F(s)\}$ along any line $x = \sigma$, with $\sigma \geq c$ converges to the function $f(t)$ that is independent of σ,

$$f(t) = \mathscr{L}_t^{-1}\{F(s)\}$$

whose Laplace transform is $F(s)$,

$$F(s) = \mathscr{L}\{f(t)\}, \quad \text{Re}(s) \geq c\,.$$

In addition, the function $f(t)$ is continuous for $t > 0$ and $f(0) = 0$, and $f(t)$ is of the order $O(e^{ct})$ for all $t > 0$.

Suppose that there are two transformable functions $f_1(t)$ and $f_2(t)$ that have the same transforms

$$\mathscr{L}\{f_1(t)\} = \mathscr{L}\{f_2(t)\} = F(s)\,.$$

The difference between the two functions is written $\phi(t)$

$$\phi(t) = f_1(t) - f_2(t)$$

where $\phi(t)$ is a transformable function. Thus,

$$\mathscr{L}\{\phi(t)\} = F(s) - F(s) = 0\,.$$

Additionally,

$$\phi(t) = \mathscr{L}_t^{-1}\{0\} = 0\,, \quad t > 0\,.$$

Therefore, this requires that $f_1(t) = f_2(t)$. The result shows that it is not possible to find two different functions by using two different values of σ in the inversion integral. This conclusion can be expressed as follows:

Theorem 5.6.2

Only a single function $f(t)$ that is sectionally continuous, of exponential order, and with a mean value at each point of discontinuity, corresponds to a given transform $F(s)$.

5.7 Applications to Partial Differential Equations

The Laplace transformations can be very useful in the solution of partial differential equations. A basic class of partial differential equations is applicable to a wide range of problems. However, the form of the solution in a given case is critically dependent on the boundary conditions that apply in any particular case. In consequence, the steps in the solution often will call on many different mathematical techniques. Generally, in such problems the resulting inverse transforms of more complicated functions of s occur than those for most linear systems problems. Often the inversion integral is useful in the solution of such problems. The following examples will demonstrate the approach to typical problems.

Example 5.7.1

Solve the typical heat conduction equation

$$\frac{\partial^2 \varphi}{\partial x^2} = \frac{\partial \varphi}{\partial t}, \quad 0 < x < \infty, \quad t \geq 0 \tag{7.1}$$

subject to the conditions

C-1. $\varphi(x, 0) = f(x), \quad t = 0$

C-2. $\dfrac{\partial \varphi}{\partial x} = 0, \ \varphi(x, t) = 0 \quad x = 0.$

Solution

Multiply both sides of (7.1) by $e^{-sx}\,dx$ and integrate from 0 to ∞.

$$\Phi(s, t) = \int_0^\infty e^{-sx} \varphi(x, t)\, dx.$$

Also

$$\int_0^\infty \frac{\partial^2 \varphi}{\partial x^2} e^{-sx} dx = s^2 \Phi(s, t) - s\varphi(0+) - \frac{\partial \varphi}{\partial x}(0+).$$

Equation (7.1) thus transforms, subject to C-2 and zero boundary conditions, to

$$\frac{d\Phi}{dt} - s^2 \Phi = 0.$$

The solution to this equation is

$$\Phi = Ae^{s^2 t}.$$

By an application of condition C-1, in transformed form, we have

$$\Phi = A = \int_0^\infty f(\lambda) e^{-s\lambda} d\lambda.$$

The solution, subject to C-1, is then

$$\Phi(s, t) = e^{+s^2 t} \int_0^\infty f(\lambda) e^{-s\lambda} d\lambda.$$

Now apply the inversion integral to write the function in terms of x from s,

$$\varphi(x, t) = \frac{1}{2\pi j} \int_{-\infty}^\infty e^{+s^2 t} \left[\int_0^\infty f(\lambda) e^{-s\lambda} d\lambda \right] e^{sx} ds$$

$$= \frac{1}{2\pi j} \int_{-\infty}^\infty f(\lambda)\, d\lambda \int_0^\infty e^{s^2 t - s\lambda + sx} ds.$$

Note that we can write

$$s^2 t - s(x-\lambda) = \left\{ s\sqrt{t} - \frac{(x-\lambda)}{2\sqrt{t}} \right\}^2 - \frac{(x-\lambda)^2}{4t}.$$

Also write

$$s\sqrt{t} - \frac{(x-\lambda)}{2\sqrt{t}} = u.$$

Then

$$\varphi(x,t) = \frac{1}{2\pi j}\int_{-\infty}^{\infty} f(\lambda) \exp - \left[\frac{(x-\lambda)^2}{4t} \right] d\lambda \int_0^{\infty} e^{-u^2} \frac{du}{\sqrt{t}}.$$

But the integral

$$\int_0^{\infty} e^{-u^2} du = \sqrt{\pi}.$$

Thus, the final solution is

$$\varphi(x,t) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} f(\lambda) e^{-\frac{(x-\lambda)^2}{4t}} d\lambda.$$

Example 5.7.2

A semi-infinite medium, initially at temperature $\varphi = 0$ throughout the medium, has the face $x = 0$ maintained at temperature φ_0. Determine the temperature at any point of the medium at any subsequent time.

Solution

The controlling equation for this problem is

$$\frac{\partial^2 \varphi}{\partial x^2} = \frac{1}{K}\frac{\partial \varphi}{\partial t} \tag{7.2}$$

with the boundary conditions:

a. $\varphi = \varphi_0$ at $x = 0$, $t > 0$
b. $\varphi = 0$ at $t = 0$, $x > 0$.

To proceed, multiply both sides of equation (7.2) by $e^{-st}\, dt$ and integrate from 0 to ∞. The transformed form of equation (7.2) is

$$\frac{d^2\Phi}{dx^2} - \left(\frac{s}{K}\Phi \right) = 0, \quad K > 0.$$

The solution of this differential equation is

$$\Phi = Ae^{-x\sqrt{s/K}} + Be^{x\sqrt{s/K}}.$$

But Φ must be finite or zero for infinite x; therefore, $B = 0$ and

$$\Phi(s, x) = Ae^{-\sqrt{\frac{s}{K}}x}.$$

Apply boundary condition (a) in transformed form, namely

$$\Phi(0, s) = \int_0^\infty e^{-st}\varphi_0 dt = \frac{\varphi_0}{s} \quad \text{for } x = 0.$$

Therefore,

$$A = \frac{\varphi_0}{s}$$

and the solution in Laplace transformed form is

$$\Phi(s, x) = \frac{\varphi_0}{s} e^{-\sqrt{\frac{s}{K}}x}. \tag{7.3}$$

To find $\varphi(x, t)$ requires that we find the inverse transform of this expression. This requires evaluating the inversion integral

$$\varphi(x, t) = \frac{\varphi_0}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} \frac{e^{-x\sqrt{\frac{s}{K}}} e^{st}}{s} ds. \tag{7.4}$$

This integral has a branch point at the origin (see Figure 5.13). To carry out the integration, we select a path such as that shown (see also Figure 5.11). The integral in (7.4) is written

$$\varphi(x, t) = \frac{\varphi_0}{2\pi j}\left[\int_{BC} + \int_{\Gamma_2} + \int_{l-} + \int_{\gamma} + \int_{l+} + \int_{\Gamma_3} + \int_{FG}\right].$$

As in Example 5.6.2

$$\int_{\Gamma_2} = \int_{\Gamma_3} = \int_{BC} = \int_{FG} = 0.$$

For the segments

$$\int_{\ell-}, \text{ let } s = \rho e^{j\pi} \quad \text{and for } \int_{\ell+}, \text{ let } s = \rho e^{j\pi}.$$

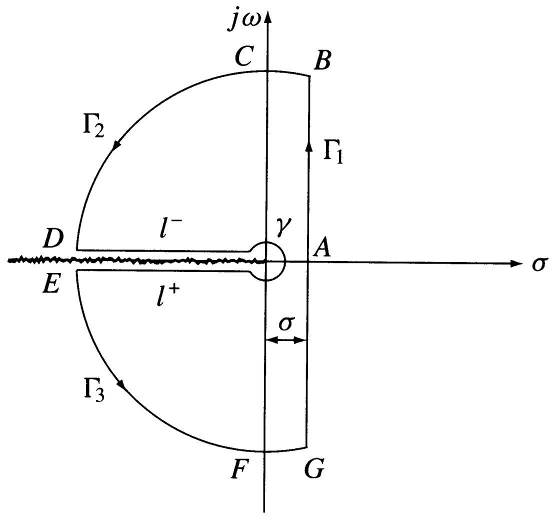


FIGURE 5.13 The path of integration.

Then for $\ell-$ and $\ell+$, writing this sum I_ℓ,

$$I_\ell = \frac{1}{2\pi j}\int_0^\infty e^{-st}\left[e^{jx\sqrt{s/K}} - e^{-jx\sqrt{s/K}}\right]\frac{ds}{s} = -\frac{1}{\pi}\int_0^\infty e^{-st}\sin x\sqrt{\frac{s}{K}}\,\frac{ds}{s}.$$

Write

$$u = \sqrt{\frac{s}{K}} \quad s = ku^2, \quad ds = 2ku\,du.$$

Then we have

$$I_\ell = -\frac{2}{\pi}\int_0^\infty e^{-Ku^2 t}\sin ux\frac{du}{u}.$$

This is a known integral that can be written

$$I_l = -\frac{2}{\sqrt{\pi}}\int_0^{\frac{x}{2\sqrt{Kt}}} e^{-u^2}\,du.$$

Finally, consider the integral over the hook,

$$I_\gamma = \frac{1}{2\pi j}\int_\gamma e^{st}\frac{e^{x\sqrt{s/K}}}{s}\,ds.$$

Let us write

$$s = re^{j\theta}, \quad ds = jre^{j\theta}d\theta, \quad \frac{ds}{s} = j\theta,$$

then

$$I_\gamma = \frac{j}{2\pi j}\int e^{t\,re^{j\theta}} e^{x\sqrt{r/K}\,e^{j\theta/2}}\,d\theta\,.$$

For $r \to 0$, $I_\gamma = \frac{j2\pi}{2\pi j} = \frac{2\pi j}{2\pi j}$, then $I_\gamma = 1$. Hence, the sum of the integrals in (7.3) becomes

$$\varphi(t) = \varphi_0\left[1 - \frac{2}{\sqrt{\pi}}\int_0^{\frac{x}{2\sqrt{Kt}}} e^{-u^2}\,du = \varphi_0\left[1 - \operatorname{erf}\frac{x}{2\sqrt{Kt}}\right].\right. \tag{7.5}$$

Example 5.7.3

A finite medium of length l is at initial temperature φ_0. There is no heat flow across the boundary at $x = 0$, and the face at $x = l$ is then kept at φ_1 (see Figure 5.14). Determine the temperature $\varphi(t)$.

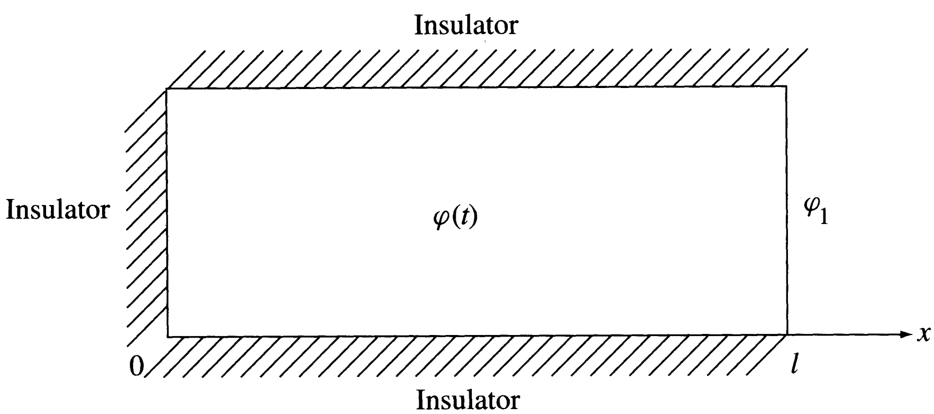


FIGURE 5.14 Details for Example 5.7.3.

Solution
Here we have to solve

$$\frac{\partial^2\varphi}{\partial x^2} = \frac{1}{k}\frac{\partial\varphi}{\partial t}$$

subject to the boundary conditions:

a. $\varphi = \varphi_0 \quad t = 0 \quad 0 \le x \le l$
b. $\varphi = \varphi_1 \quad t > 0 \quad x = l$
c. $\frac{\partial\varphi}{\partial x} = 0 \quad t > 0 \quad x = 0.$

Upon Laplace transforming the controlling differential equation, we obtain

$$\frac{d^2\Phi}{dx^2} - \frac{s}{k}\Phi = 0\,.$$

The solution is

$$\Phi = A'e^{-x\sqrt{\frac{s}{k}}} + B'e^{x\sqrt{\frac{s}{k}}} = A\cosh x\sqrt{\frac{s}{k}} + B\sinh x\sqrt{\frac{s}{k}}\,.$$

By condition *c*

$$\frac{d\Phi}{dx} = 0 \quad x = 0 \quad t > 0\,.$$

This imposes the requirement that $B = 0$, so that

$$\Phi = A\cosh x\sqrt{\frac{s}{k}}\,.$$

Now condition b is imposed. This requires that

$$\frac{\varphi_1}{s} = A\cosh l\sqrt{\frac{s}{k}}\,.$$

Thus, by b and c

$$\Phi = \varphi_1 \frac{\cosh x\sqrt{\frac{s}{k}}}{s\cosh l\sqrt{\frac{s}{k}}}\,.$$

Now, to satisfy c we have

$$\Phi = \frac{\varphi_0}{s} - \frac{\varphi_0}{s}\frac{\cosh x\sqrt{\frac{s}{k}}}{\cosh l\sqrt{\frac{s}{k}}}\,.$$

Thus, the final form of the Laplace transformed equation that satisfies all conditions of the problem is

$$\Phi = \frac{\varphi_0}{s} + \frac{\varphi_1 - \varphi_0}{s}\frac{\cosh x\sqrt{\frac{s}{k}}}{\cosh l\sqrt{\frac{s}{k}}}\,.$$

To find the expression for $\varphi(x, t)$, we must invert this expression. That is,

$$\varphi(x, t) = \varphi_0 + \frac{\varphi_1 - \varphi_0}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} e^{st}\frac{\cosh x\sqrt{\frac{s}{k}}}{\cosh l\sqrt{\frac{s}{k}}}\frac{ds}{s}\,. \qquad (7.6)$$

The integrand is a single valued function of s with poles at $s = 0$ and $s = -k\left(\frac{2n-1}{2}\right)^2 \frac{\pi^2}{l^2}$, $n = 1, 2, \ldots$.

We select the path of integration that is shown in Figure 5.15. But the inversion integral over the path $BCA(=\Gamma) = 0$. Thus, the inversion integral becomes

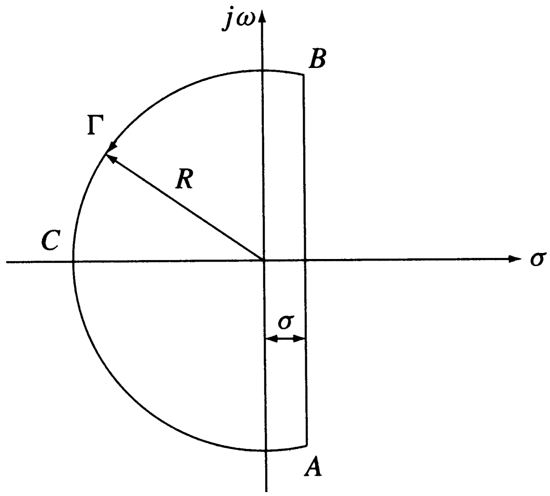


FIGURE 5.15 The path of integration for Example 5.7.3.

$$\frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} e^{st}\frac{\cosh x\sqrt{\frac{s}{k}}}{\cosh l\sqrt{\frac{s}{k}}}\frac{ds}{s}.$$

By an application of the Cauchy integral theorem, we require the residues of the integrand at its poles. There results

$$\mathrm{Res}\big|_{s=0} = 1$$

$$\mathrm{Re}\,s\Big|_{s=-k\left(n-\frac{1}{2}\right)^2\frac{\pi^2}{l^2}} = \frac{e^{-k\left(n-\frac{1}{2}\right)^2\frac{\pi^2}{l^2}}\cosh j\left(n-\frac{1}{2}\right)\frac{\pi x}{l}}{\left[s\frac{d}{ds}\left\{\cosh l\sqrt{\frac{s}{k}}\right\}_{s=-k\left(n-\frac{1}{2}\right)^2\frac{\pi^2}{l^2}}\right]}.$$

Combine these with (7.5) to write finally

$$\varphi(x,t) = \varphi_0 + \frac{4(\varphi_1-\varphi_0)}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1} l^{-k\left(n-\frac{1}{2}\right)^2\pi^2/l^2}\cos\left[\left(n-\frac{1}{2}\right)\pi x/l\right]. \tag{7.7}$$

Example 5.7.4

A circular cylinder of radius a is initially at temperature zero. The surface is then maintained at temperature φ_0. Determine the temperature of the cylinder at any subsequent time t.

Solution

The heat conduction equation in radial form is

$$\frac{\partial^2\varphi}{\partial r^2}+\frac{1}{r}\frac{\partial\varphi}{\partial r}=\frac{1}{k}\frac{\partial\varphi}{\partial t},\quad 0\le r<a,\quad t>0. \tag{7.8}$$

And for this problem the system is subject to the boundary conditions

C-1. $\varphi = 0 \quad t = 0 \quad 0 \le r < a$
C-2. $\varphi = \varphi_0 \quad t > 0 \quad r = a.$

To proceed, we multiply each term in the partial differential equation by $e^{-st}dt$ and integrate. We write

$$\int_0^\infty \varphi\, e^{-st}dt=\Phi(r,s)$$

Then (7.7) transforms to

$$k\left(\frac{d^2\Phi}{dr^2}+\frac{1}{r}\frac{d\Phi}{dr}\right)-s\Phi=0,$$

which we write in the form

$$\frac{d^2\Phi}{dr^2}+\frac{1}{r}\frac{d\Phi}{dr}-\mu\Phi=0,\quad \mu=\sqrt{\frac{s}{k}}.$$

This is the Bessel equation of order 0 and has the solution

$$\Phi = AI_0(\mu r) + BN_0(\mu r).$$

However, the Laplace transformed form of C-1 when $z = 0$ imposes the condition $B = 0$ because $N_0(0)$ is not zero. Thus,

$$\Phi = AI_0(\mu r).$$

The boundary condition C-2 requires $\Phi(r, a) = \frac{\varphi_0}{s}$ when $r = a$, hence,

$$A=\frac{\varphi_0}{s}\frac{1}{I(\mu a)}$$

so that

$$\Phi = \frac{\varphi_0}{s}\frac{I_0(\mu r)}{I_0(\mu a)}.$$

To find the function $\varphi(r, t)$ requires that we invert this function. By an application of the inversion integral, we write

$$\varphi(r,t) = \frac{\varphi_0}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} e^{\lambda t}\frac{I_0(\xi r)}{I_0(\xi a)}\frac{d\lambda}{\lambda}, \quad \xi = \sqrt{\frac{\lambda}{k}}. \tag{7.9}$$

Note that $I_0(\xi r)/I_0(\xi a)$ is a single-valued function of λ. To evaluate this integral, we choose as the path for this integration that shown in Figure 5.16. The poles of this function are at $\lambda = 0$ and at the roots of the Bessel function $J_0(\xi a)$ $(= I_0(j\xi a))$; these occur when $J_0(\xi a) = 0$, with the roots for $J_0(\xi a) = 0$, namely $\lambda = -k\xi_1^2, -k\xi_2^2, \ldots$. The approximations for $I_0(\xi r)$ and $I_0(\xi a)$ show that when $n \to \infty$ the integral over the path BCA tends to zero. The resultant value of the integral is written in terms of the residues at zero and when $\lambda = k\xi_n^2$. These are

$$\text{Res}\Big|_{=0} = 1$$

$$\text{Res}\Big|_{k\xi_n^2} = \frac{\lambda dI_0(\xi a)}{d\lambda}\Bigg|_{k\xi_n^2}.$$

Therefore,

$$\varphi(r,t) = \varphi_0\left[1 + \sum_n e^{-k\xi_n^2 t}\frac{J_0(\xi_n r)}{\lambda \frac{d}{d\lambda} I_0(\xi a)\Big|_{\lambda = k\xi_n^2}}\right].$$

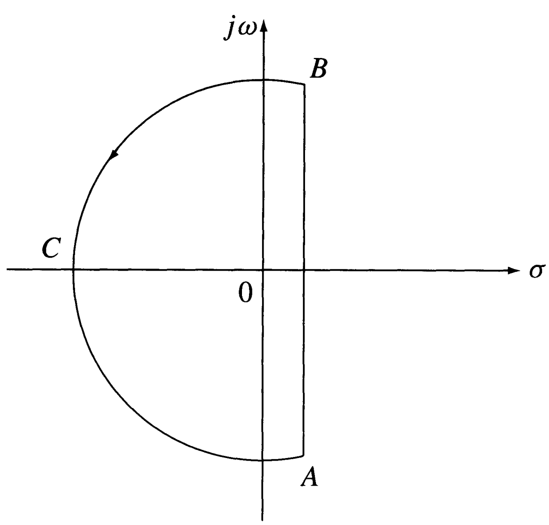


FIGURE 5.16 The path of integration for Example 5.7.4.

Further, $\lambda \frac{d}{d\lambda} I_0(\xi_n a) = \frac{1}{2} \xi a I_0^{(1)}(\xi a)$. Hence, finally,

$$\varphi(t) = \varphi_0 \left[1 + \frac{2}{a} \sum_{n=1}^{\infty} e^{-k\xi_n^2 t} \frac{J_0(\xi_n r)}{\xi_n J_0^{(1)}(\xi_n a)} \right]. \tag{7.10}$$

Example 5.7.5

A semi-infinite stretched string is fixed at each end. It is given an initial transverse displacement and then released. Determine the subsequent motion of the string.

Solution

This requires solving the wave equation

$$a^2 \frac{\partial^2 \varphi}{\partial x^2} = \frac{\partial^2 \varphi}{\partial t^2} \tag{7.11}$$

subject to the conditions

C-1. $\varphi(x, 0) = f(x) \quad t = 0, \varphi(0, t) = 0 \quad t > 0$
C-2. $\lim_{x \to \infty} \varphi(x, t) = 0$.

To proceed, multiply both sides of (7.11) by $e^{-st}dt$ and integrate. The result is the Laplace-transformed equation

$$a^2 \frac{d^2\Phi}{dx^2} = s^2\Phi - s\varphi(0+), \quad x > 0. \tag{7.12}$$

C-1. $\Phi(0, s) = 0$
C-2. $\lim_{x \to \infty} \Phi(x, s) = 0$.

To solve (7.12) we will carry out a second Laplace transform, but this with respect to x, that is $\mathscr{L}\{\Phi(x, s)\} = N(z, s)$. Thus,

$$N(z, s) = \int_0^{\infty} \Phi(x, s) e^{-zx} dx.$$

Apply this transformation to both members of (7.12) subject to $\Phi(0, s) = 0$. The result is

$$s^2 N(z, s) - s\Phi(z) = a^2 \left[z^2 N(z, s) - \frac{\partial \Phi}{\partial x}(0, s) \right], \quad \Phi(z) = \mathscr{L}\{\varphi_0\}.$$

We denote $\frac{\partial \Phi}{\partial x}(0, s)$ by C. Then the solution of this equation is

$$N(z, s) = \frac{C}{z^2 - \frac{s^2}{a^2}} - \frac{s}{a^2} \Phi(z) \frac{1}{z^2 - \frac{s^2}{a^2}}.$$

The inverse transformation with respect to z is, employing convolution.

$$\Phi(x,s)=\frac{aC}{s}\sinh\frac{sx}{a}-\frac{1}{a}\int_0^x \varphi(\xi)\sinh\frac{s}{a}(x-\xi)\,d\xi .$$

To satisfy the condition $\lim_{x\to\infty}\Phi(x,s)=0$ requires that the sinh terms be replaced by their exponential forms. Thus, the factors

$$\sinh\frac{sx}{a}\to\frac{1}{2},\quad \sinh\frac{s}{a}(x-\xi)\to\frac{e^{-s\xi/a}}{2},\quad x\to\infty .$$

Then we have the expression

$$\Phi(x,s)=\frac{aC}{2s}-\frac{1}{2a}\int_0^x \varphi(\xi)e^{-s\xi/a}d\xi .$$

But for this function to be zero for $x\to\infty$ requires that

$$\frac{aC}{s}=\frac{1}{a}\int_0^\infty \varphi(\xi)e^{-s\xi/a}d\xi ,\quad x\to\infty .$$

Combine this result with $\Phi(x,s)$ to get

$$2a\Phi(x,s)=\int_0^\infty \varphi(\xi)e^{-s(\xi-x)/a}d\xi-\int_0^\infty \varphi(\xi)e^{-s(x+\xi)/a}d\xi+\int_0^x \varphi(\xi)e^{-s(x-\xi)/a}d\xi .$$

Each integral in this expression is integrated by parts. Here we write

$$u=\varphi(\xi),\quad du=\varphi^{(1)}(\xi)d\xi;\quad dv=e^{-\frac{s(\xi-x)}{a}}d\xi,\quad v=\frac{a}{s}e^{-s(\xi-k)/a} .$$

The resulting integrations lead to

$$\Phi(x,s)=\frac{1}{s}\varphi(x)+\frac{1}{2s}\int_x^\infty \varphi^{(1)}(\xi)e^{-\frac{s(\xi-x)}{a}}d\xi-\frac{1}{2s}\int_0^\infty \varphi^{(1)}(\xi)e^{-\frac{s(x+\xi)}{a}}d\xi$$

$$-\frac{1}{2s}\int_0^\infty \varphi^{(1)}(\xi)e^{-\frac{s(x-\xi)}{a}}d\xi .$$

We note by entry 61, Table 5.1 that

$$\mathscr{L}^{-1}\left\{\frac{1}{s}e^{\frac{-s(\xi-x)}{a}}\right\}=1 \qquad \text{when } at>\xi-x$$

$$=0 \qquad \text{when } at<\xi-x .$$

This function of ξ vanishes except when $\xi\le x+at$. Thus,

$$\mathcal{L}^{-1}\left\{\frac{1}{2}\int_x^\infty \varphi^{(1)}(\xi)e^{-s(\xi+x)/a}d\xi\right\} = \frac{1}{2}\int_x^{x+at}\varphi^{(1)}(\xi)d\xi = \frac{1}{2}\varphi(x+at) - \frac{1}{2}\varphi(x).$$

Proceed in the same way for the term

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{2}\int_0^\infty \varphi^{(1)}(\xi)e^{-s(x+\xi)/a}d\xi\right\} &= 1 \qquad \text{when } at > x+\xi \\ &= 0 \qquad \text{when } at < x+\xi.\end{aligned}$$

Thus, the second term becomes

$$\mathcal{L}^{-1}\left\{\frac{1}{2}\int_x^\infty \varphi^{(1)}(\xi)e^{-s(x+\xi)/a}d\xi\right\} = \frac{1}{2}\int_x^{x+at}\varphi^{(1)}(\xi)d\xi = \frac{1}{2}\varphi(x-at).$$

The final term becomes

$$\begin{aligned}&-\frac{1}{2}\varphi(x) && \text{when } at > x \\ &-\frac{1}{2}\varphi(x) + \frac{1}{2}\varphi(x-at) && \text{when } at < x.\end{aligned}$$

The final result is

$$\begin{aligned}\varphi(x,t) &= \frac{1}{2}\left[f(at+x) - f(at-x)\right] \quad \text{when } t > x/a \\ &= \frac{1}{2}\left[f(x+at) + f(x-at)\right] \quad \text{when } t < x/a.\end{aligned} \tag{7.13}$$

Example 5.7.6

A stretched string of length l is fixed at each end as shown in Figure 5.17. It is plucked at the midpoint and then released at $t = 0$. The displacement is b. Find the subsequent motion.

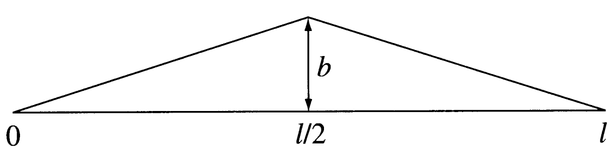


FIGURE 5.17 A stretched string plucked at its midpoint.

Solution

This problem requires the solution of

$$\frac{\partial^2 y}{\partial y^2} = c^2\frac{\partial^2 y}{\partial t^2}, \quad 0 < y < l, \quad t > 0 \tag{7.14}$$

subject to

$$
\left.
\begin{array}{lll}
1. & y=\dfrac{2bx}{l} & 0<x<l/2 \\
2. & y=\dfrac{2b}{l}(l-x) & \dfrac{1}{2}<x<l
\end{array}
\right\} t=0
$$

$$
\begin{array}{lllll}
3. & \dfrac{\partial y}{\partial t}=0 & 0<x<l & t=0 & \\
4. & y=0 & x=0; & x=l & t>0.
\end{array}
$$

To proceed, multiply (7.13) by $e^{-st}dt$ and integrate in t. This yields

$$s^2Y-sy(0)=c^2\frac{d^2Y}{dx^2}$$

or

$$c^2\frac{d^2Y}{dx^2}-s^2Y=sy(0)=-sf(x) \tag{7.15}$$

subject to $Y(0, s) = Y(l, s) = 0$. To solve this equation, we proceed as in Example 5.7.5; that is, we apply a transformation on x, namely $\mathscr{L}\{Y(x, s)\} = N(z, s)$. Thus,

$$s^2N(z,s)-sY(0)=c^2\left[z^2N(z,s)-\frac{y(0,s)}{x}\right].$$

This equation yields, writing $sY(0)$ as $\Phi(x, s)$,

$$N(z,s)=\frac{\Phi(z,s)}{z^2-\dfrac{s^2}{c^2}}$$

The inverse transform is

$$Y(x,s)=\mathscr{L}^{-1}\{N(x,s)\}=\frac{\Phi(x,s)}{c\sinh\dfrac{s}{c}}$$

where

$$\Phi(x,s)=\sinh\frac{(l-x)s}{c}\int_0^x y(\xi)\sinh\frac{\xi s}{c}d\xi+\sinh\frac{xs}{c}\int_0^l y(\xi)\sinh\frac{(l-\xi)s}{c}d\xi.$$

Combine these integrals with the known form of $f(x)$ in C-1 and C-2. Upon carrying out the integrations, the resulting forms become, with $k=\frac{s}{c}$,

$$\frac{lY}{2b} = \frac{x}{s} - \frac{c}{s^2}\frac{\sinh kx}{\cosh \frac{kl}{2}}, \qquad 0 \le x \le l/2$$

$$\frac{lY}{2b} = \frac{(l-x)}{s} - \frac{c \sinh k(l-x)}{s^2 \cosh \frac{kl}{2}}, \qquad \frac{l}{2} \le x \le l.$$

To find $y(t)$, we must invert these expressions. Note that symmetry exists and so we need consider only the first term. We use the inversion integral on the term $\frac{1}{s^2}\frac{\sinh kx}{\cosh \frac{kl}{2}}$. Thus, we consider the integral

$$I = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} e^{\lambda b} \frac{\sinh \frac{x\lambda}{c}}{\cosh \frac{x\lambda}{2c}} \frac{d\lambda}{\lambda^2}.$$

We choose the path in the λ-plane as shown in Figure 5.18. The value of the integral over path Γ is zero. Thus, the value of the integral is given in terms of the residues. These occur at $\lambda = 0$ and at the values for which $\cosh \frac{\lambda}{2c} = 0$, which exist where

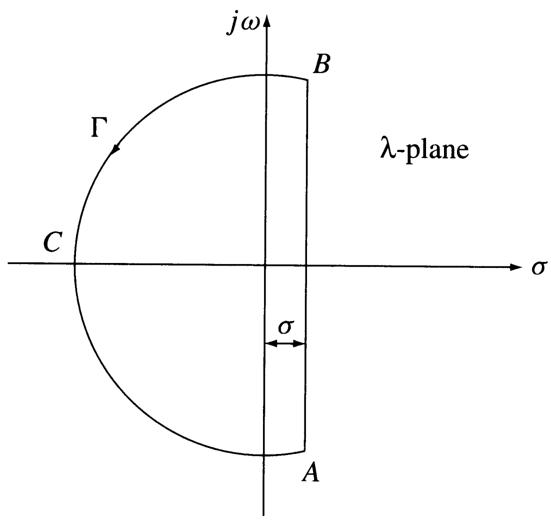


FIGURE 5.18 The path of integration for Example 5.7.6.

$$\frac{\lambda}{2c} = j\frac{2n-1}{l}\frac{\pi}{2} \quad \text{or} \quad \lambda = \pm j\frac{2n-1}{l}\pi c.$$

Thus, we have, by the theory of residues

$$\text{Res}\Big|_{\lambda=0} = \frac{x}{c}$$

$$\text{Res}\Big|_{j\frac{2n-1}{l}\pi c} = e^{j(2n-1)}\,\frac{\pi c x}{l}\,\frac{\sinh j(2n-1)\dfrac{\pi x}{l}}{\dfrac{d}{d\lambda}\left[\lambda^2 \cosh\dfrac{\lambda l}{2c}\right]_{j\frac{(2n-1)}{l}\pi c}}.$$

These poles lead to

$$= (-1)^n \frac{2l}{\pi^2 c}\,\frac{\sin(2n-1)\dfrac{\pi x}{l}}{(2n-1)^2}\,e^{j(2n-1)\frac{\pi c}{l}t}.$$

Thus, the poles at $\pm j(2n-1)\,\dfrac{\pi c}{l}$ lead to

$$= (-1)^n \frac{4l}{\pi^2 c}\,\frac{\sin(2n-1)\dfrac{\pi x}{l}\cos(2n-1)\dfrac{\pi c}{l}t}{(2n-1)^2}.$$

Then we have

$$\frac{\ell y}{2b} = x - \left\{x + \frac{4l}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^2}\sin(2n-1)\frac{\pi x}{l}\cos(2n-1)\frac{\pi c t}{l}\right\}$$

so that finally

$$y = \frac{8b}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^2}\sin(2n-1)\frac{\pi x}{l}\cos(2n-1)\frac{\pi c t}{l}, \quad 0 \le x \le \frac{l}{2}.$$

For the string for which $\dfrac{l}{2} \le x < l$, the corresponding expression is the same except that $(l - x)$ replaces x.

Note that this equation can be written, with $\eta = (2n-1)\,\dfrac{\pi}{l}$,

$$\sin \eta x \cos \eta c t = \frac{\sin\eta(x - ct) + \sin\eta(x + ct)}{2},$$

which shows the traveling wave nature of the solution.

5.8 The Bilateral or Two-Sided Laplace Transform

In Section 5.1 we discussed the fact that the region of absolute convergence of the unilateral or one-sided Laplace transform is the region to the left of the abscissa of convergence. The situation for the two-sided Laplace transform is rather different; the region of convergence must be specified if we wish to invert a function $F(s)$ that was obtained using the bilateral Laplace transform. This requirement is necessary because different time signals might have the same Laplace transform but different regions of absolute convergence.

To establish the region of convergence, write the bilateral Laplace transform in the form

$$F_2(s) = \int_{-\infty}^{\infty} e^{-st} f(t)dt = \int_{0}^{\infty} e^{-st} f(t)dt + \int_{-\infty}^{0} e^{-st} f(t)dt. \qquad (8.1)$$

If the function $f(t)$ is of exponential order ($e^{\sigma 1 t}$), then the region of convergence for $t > 0$ is $\mathrm{Re}(s) > \sigma_1$. If the function $f(t)$ for $t < 0$ is of exponential order $\exp(\sigma_2 t)$, then the region of convergence is $\mathrm{Re}(s) < \sigma_2$. Hence, the function $F_2(s)$ exists and its analytic in the vertical strip defined by $\sigma_1 < \mathrm{Re}(s) < \sigma_2$, provided, of course, that $\sigma_1 < \sigma_2$. If $\sigma_2 > \sigma_1$, no region of convergence would exist and the inversion process could not be performed. This region of convergence is shown in Figure 5.19.

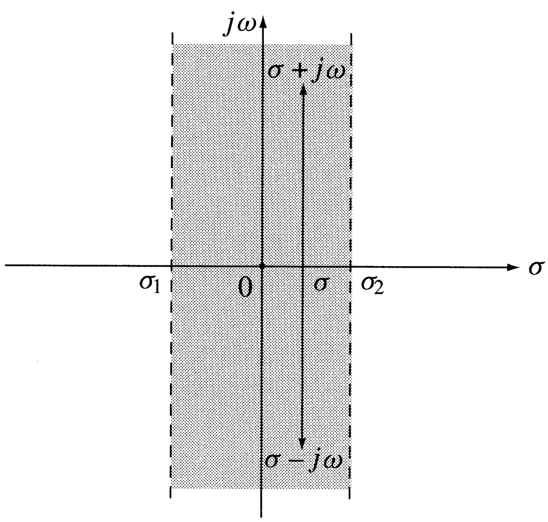


FIGURE 5.19 Region of convergence for the bilateral transform.

Example 5.8.1

Find the bilateral Laplace transform of the signals $f(t) = e^{-at}u(t)$ and $f(t) = -e^{-at}u(t)$ and specify their regions of convergence.

Solution

Using the basic definition of the transform (8.1), we obtain

$$F_2(s) = \int_{-\infty}^{\infty} e^{-at} u(t) e^{-st} dt = \int_{0}^{\infty} e^{-(s+a)t} dt = \frac{1}{s+a}$$

and its region of convergence is $\mathrm{Re}(s) > -a$.

For the second signal,

$$F_2(s) = \int_{-\infty}^{\infty} -e^{-at} u(-t) e^{-st} dt = -\int_{-\infty}^{0} e^{-(s+a)t} dt = \frac{1}{s+a}$$

and its region of convergence is Re(s) < $-a$.

Clearly, the knowledge of the region of convergence is necessary to find the time functions unambiguously.

Example 5.8.2

Find the function $f(t)$, if its Laplace transform is given by

$$F_2(s) = \frac{3}{(s-4)(s+1)(s+2)}, \quad -2 < \text{Re}(s) < -1 .$$

Solution

The region of convergence and the paths of integration are shown in Figure 5.20. For $t > 0$ we close the contour to the left and we obtain

$$f(t) = \left.\frac{3e^{st}}{(s-4)(s+1)}\right|_{s=-2} = \frac{1}{2}e^{-2t}, \quad t > 0 .$$

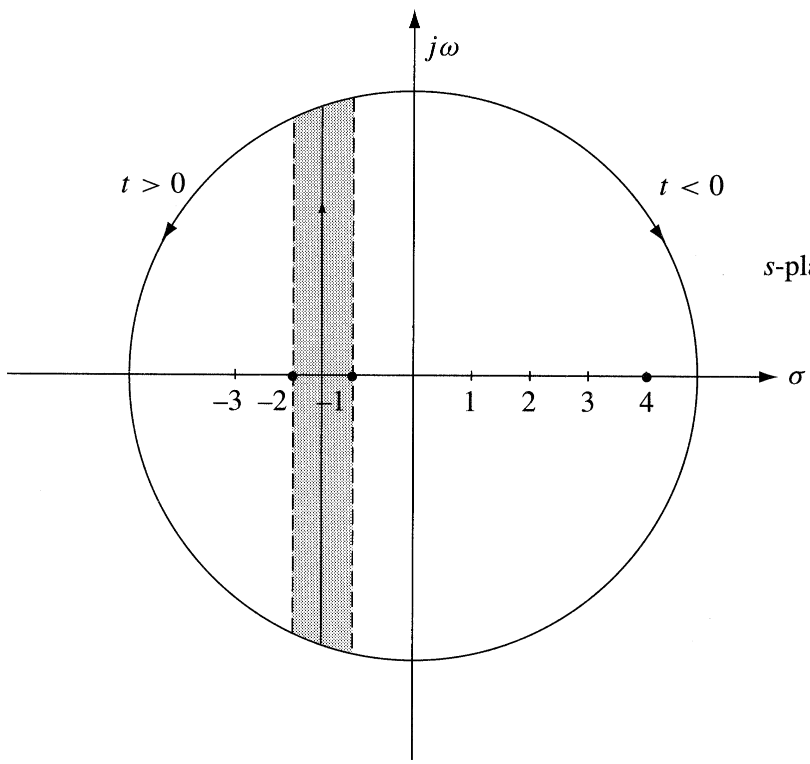


FIGURE 5.20 Illustrating Example 5.8.2.

For $t < 0$, the contour closes to the right and now

$$f(t) = \left.\frac{3e^{st}}{(s-4)(s+2)}\right|_{s=-1} + \left.\frac{3e^{st}}{(s+1)(s+2)}\right|_{s=4} = -\frac{3}{5}e^{-t} + \frac{e^{4t}}{10}, \quad t < 0 .$$

Appendix

TABLE 5.1 Laplace Transform Pairs

	$F(s)$	$f(t)$
1	s^n	$\delta^{(n)}(t)$ n^{th} derivative of the delta function
2	s	$\frac{d\delta(t)}{dt}$
3	1	$\delta(t)$
4	$\frac{1}{s}$	1
5	$\frac{1}{s^2}$	t
6	$\frac{1}{s^n}\ (n=1,2,\cdots)$	$\frac{t^{n-1}}{(n-1)!}$
7	$\frac{1}{\sqrt{s}}$	$\frac{1}{\sqrt{\pi t}}$
8	$s^{-3/2}$	$2\sqrt{\frac{t}{\pi}}$
9	$s^{-[n+(1/2)]}\ (n=1,2,\cdots)$	$\frac{2^n t^{n-(1/2)}}{1\cdot 3\cdot 5\cdots(2n-1)\sqrt{\pi}}$
10	$\frac{\Gamma(k)}{s^k}\ (k\geq 0)$	t^{k-1}
11	$\frac{1}{s-a}$	e^{at}
12	$\frac{1}{(s-a)^2}$	te^{at}
13	$\frac{1}{(s-a)^n}\ (n=1,2,\cdots)$	$\frac{1}{(n-1)!}t^{n-1}e^{at}$
14	$\frac{\Gamma(k)}{(s-a)^k}\ (k\geq 0)$	$t^{k-1}e^{at}$
15	$\frac{1}{(s-a)(s-b)}$	$\frac{1}{(a-b)}(e^{at}-e^{bt})$
16	$\frac{s}{(s-a)(s-b)}$	$\frac{1}{(a-b)}(ae^{at}-be^{bt})$
17	$\frac{1}{(s-a)(s-b)(s-c)}$	$-\frac{(b-c)e^{at}+(c-a)e^{bt}+(a-b)e^{ct}}{(a-b)(b-c)(c-a)}$
18	$\frac{1}{(s+a)}$	e^{-at} valid for complex a
19	$\frac{1}{s(s+a)}$	$\frac{1}{a}(1-e^{-at})$
20	$\frac{1}{s^2(s+a)}$	$\frac{1}{a^2}(e^{-at}+at-1)$
21	$\frac{1}{s^3(s+a)}$	$\frac{1}{a^2}\left[\frac{1}{a}-t+\frac{at^2}{2}-\frac{1}{a}e^{-at}\right]$
22	$\frac{1}{(s+a)(s+b)}$	$\frac{1}{(b-a)}(e^{-at}-e^{-bt})$
23	$\frac{1}{s(s+a)(s+b)}$	$\frac{1}{ab}\left[1+\frac{1}{(a-b)}(be^{-at}-ae^{-bt})\right]$

	$F(s)$	$f(t)$
24	$\dfrac{1}{s^2(s+a)(s+b)}$	$\dfrac{1}{(ab)^2}\left[\dfrac{1}{(a-b)}(a^2e^{-bt}-b^2e^{-at})+abt-a-b\right]$
25	$\dfrac{1}{s^3(s+a)(s+b)}$	$\dfrac{1}{(ab)}\left[\dfrac{a^3-b^3}{(ab)^2(a-b)}+\dfrac{1}{2}t^2-\dfrac{(a+b)}{ab}t+\dfrac{1}{(a-b)}\left(\dfrac{b}{a^2}e^{-at}-\dfrac{a}{b^2}e^{-bt}\right)\right]$
26	$\dfrac{1}{(s+a)(s+b)(s+c)}$	$\dfrac{1}{(b-a)(c-a)}e^{-at}+\dfrac{1}{(a-b)(c-b)}e^{-bt}+\dfrac{1}{(a-c)(b-c)}e^{-ct}$
27	$\dfrac{1}{s(s+a)(s+b)(s+c)}$	$\dfrac{1}{abc}-\dfrac{1}{a(b-a)(c-a)}e^{-at}-\dfrac{1}{b(a-b)(c-b)}e^{-bt}-\dfrac{1}{c(a-c)(b-c)}e^{-ct}$
28	$\dfrac{1}{s^2(s+a)(s+b)(s+c)}$	$\dfrac{ab(ct-1)-ac-bc}{(abc)^2}+\dfrac{1}{a^2(b-a)(c-a)}e^{-at}+\dfrac{1}{b^2(a-b)(c-b)}e^{-bt}+\dfrac{1}{c^2(a-c)(b-c)}e^{-ct}$
29	$\dfrac{1}{s^3(s+a)(s+b)(s+c)}$	$\dfrac{1}{(abc)^3}[(ab+ac+bc)^2-abc(a+b+c)]-\dfrac{ab+ac+bc}{(abc)^2}t+\dfrac{1}{2abc}t^2-\dfrac{1}{a^3(b-a)(c-a)}e^{-at}-\dfrac{1}{b^3(a-b)(c-b)}e^{-bt}-\dfrac{1}{c^3(a-c)(b-c)}e^{-ct}$
30	$\dfrac{1}{s^2+a^2}$	$\dfrac{1}{a}\sin at$
31	$\dfrac{s}{s^2+a^2}$	$\cos at$
32	$\dfrac{1}{s^2-a^2}$	$\dfrac{1}{a}\sinh at$
33	$\dfrac{s}{s^2-a^2}$	$\cosh at$
34	$\dfrac{1}{s(s^2+a^2)}$	$\dfrac{1}{a^2}(1-\cos at)$
35	$\dfrac{1}{s^2(s^2+a^2)}$	$\dfrac{1}{a^3}(at-\sin at)$
36	$\dfrac{1}{(s^2+a^2)^2}$	$\dfrac{1}{2a^3}(\sin at-at\cos at)$
37	$\dfrac{s}{(s^2+a^2)^2}$	$\dfrac{t}{2a}\sin at$
38	$\dfrac{s^2}{(s^2+a^2)^2}$	$\dfrac{1}{2a}(\sin at+at\cos at)$
39	$\dfrac{s^2-a^2}{(s^2+a^2)^2}$	$t\cos at$
40	$\dfrac{s}{(s^2+a^2)(s^2+b^2)}(a^2\neq b^2)$	$\dfrac{\cos at-\cos bt}{b^2-a^2}$
41	$\dfrac{1}{(s-a)^2+b^2}$	$\dfrac{1}{b}e^{at}\sin bt$
42	$\dfrac{s-a}{(s-a)^2+b^2}$	$e^{at}\cos bt$
43	$\dfrac{1}{[(s+a)^2+b^2]^n}$	$\dfrac{-e^{-at}}{4^{n-1}b^{2n}}\sum_{r=1}^{n}\binom{2n-r-1}{n-1}(-2t)^{r-1}\dfrac{d^r}{dt^r}[\cos(bt)]$

	$F(s)$	$f(t)$
44	$\frac{s}{[(s+a)^2+b^2]^n}$	$\begin{cases} \frac{e^{-at}}{4^{n-1}b^{2n}}\left\{\sum_{r=1}^{n}\binom{2n-r-1}{n-1}(-2t)^{r-1}\frac{d^r}{dt^r}[a\cos(bt)+b\sin(bt)]\right. \\ \left.-2b\sum_{r=1}^{n-1}r\binom{2n-r-2}{n-1}(-2t)^{r-1}\frac{d^r}{dt^r}[\sin(bt)]\right\} \end{cases}$
45	$\frac{3a^2}{s^3+a^3}$	$e^{-at}-e^{(at)/2}\left(\cos\frac{at\sqrt{3}}{2}-\sqrt{3}\sin\frac{at\sqrt{3}}{2}\right)$
46	$\frac{4a^3}{s^4+4a^4}$	$\sin at \cosh at - \cos at \sinh at$
47	$\frac{s}{s^4+4a^4}$	$\frac{1}{2a^2}(\sin at \sinh at)$
48	$\frac{1}{s^4-a^4}$	$\frac{1}{2a^3}(\sinh at - \sin at)$
49	$\frac{s}{s^4-a^4}$	$\frac{1}{2a^2}(\cosh at - \cos at)$
50	$\frac{8a^3s^2}{(s^2+a^2)^3}$	$(1+a^2t^2)\sin at - \cos at$
51	$\frac{1}{s}\left(\frac{s-1}{s}\right)^n$	$L_n(t)=\frac{e^t}{n!}\frac{d^n}{dt^n}(t^n e^{-t})$ [$L_n(t)$ is the Laguerre polynomial of degree n]
52	$\frac{1}{(s+a)^n}$	$\frac{t^{(n-1)}e^{-at}}{(n-1)!}$ where n is a positive integer
53	$\frac{1}{s(s+a)^2}$	$\frac{1}{a^2}[1-e^{-at}-ate^{-at}]$
54	$\frac{1}{s^2(s+a)^2}$	$\frac{1}{a^3}[at-2+ate^{-at}+2e^{-at}]$
55	$\frac{1}{s(s+a)^3}$	$\frac{1}{a^3}\left[1-\left(\frac{1}{2}a^2t^2+at+1\right)e^{-at}\right]$
56	$\frac{1}{(s+a)(s+b)^2}$	$\frac{1}{(a-b)^2}\{e^{-at}+[(a-b)t-1]e^{-bt}\}$
57	$\frac{1}{s(s+a)(s+b)^2}$	$\frac{1}{ab^2}-\frac{1}{a(a-b)^2}e^{-at}-\left[\frac{1}{b(a-b)}t+\frac{a-2b}{b^2(a-b)^2}\right]e^{-bt}$
58	$\frac{1}{s^2(s+a)(s+b)^2}$	$\frac{1}{a^2(a-b)^2}e^{-at}+\frac{1}{ab^2}\left(t-\frac{1}{a}-\frac{2}{b}\right)+\left[\frac{1}{b^2(a-b)}t+\frac{2(a-b)-b}{b^3(a-b)^2}\right]e^{-bt}$
59	$\frac{1}{(s+a)(s+b)(s+c)^2}$	$\begin{cases} \left[\frac{1}{(c-b)(c-a)}t+\frac{2c-a-b}{(c-a)^2(c-b)^2}\right]e^{-ct} \\ +\frac{1}{(b-a)(c-a)^2}e^{-at}+\frac{1}{(a-b)(c-b)^2}e^{-bt} \end{cases}$
60	$\frac{1}{(s+a)(s^2+\omega^2)}$	$\frac{1}{a^2+\omega^2}e^{-at}+\frac{1}{\omega\sqrt{a^2+\omega^2}}\sin(\omega t-\phi);\quad \phi=\tan^{-1}\left(\frac{\omega}{a}\right)$
61	$\frac{1}{s(s+a)(s^2+\omega^2)}$	$\frac{1}{a\omega^2}-\frac{1}{a^2+\omega^2}\left(\frac{1}{\omega}\sin\omega t+\frac{a}{\omega^2}\cos\omega t+\frac{1}{a}e^{-at}\right)$

	$F(s)$	$f(t)$
62	$\frac{1}{s^2(s+a)(s^2+\omega^2)}$	$\begin{cases} \frac{1}{a\omega^2}t - \frac{1}{a^2\omega^2} + \frac{1}{a^2(a^2+\omega^2)}e^{-at} \\ + \frac{1}{\omega^3\sqrt{a^2+\omega^2}}\cos(\omega t+\phi); \quad \phi = \tan^{-1}\left(\frac{a}{\omega}\right) \end{cases}$
63	$\frac{1}{[(s+a)^2+\omega^2]^2}$	$\frac{1}{2\omega^3}e^{-at}[\sin\omega t - \omega t\cos\omega t]$
64	$\frac{1}{s^2-a^2}$	$\frac{1}{a}\sinh at$
65	$\frac{1}{s^2(s^2-a^2)}$	$\frac{1}{a^3}\sinh at - \frac{1}{a^2}t$
66	$\frac{1}{s^3(s^2-a^2)}$	$\frac{1}{a^4}(\cosh at - 1) - \frac{1}{2a^2}t^2$
67	$\frac{1}{s^3+a^3}$	$\frac{1}{3a^2}\left[e^{-at} - e^{\frac{a}{2}t}\left(\cos\frac{\sqrt{3}}{2}at - \sqrt{3}\sin\frac{\sqrt{3}}{2}at\right)\right]$
68	$\frac{1}{s^4+4a^4}$	$\frac{1}{4a^3}(\sin at\cosh at - \cos at\sinh at)$
69	$\frac{1}{s^4-a^4}$	$\frac{1}{2a^3}(\sinh at - \sin at)$
70	$\frac{1}{[(s+a)^2-\omega^2]}$	$\frac{1}{\omega}e^{-at}\sinh\omega t$
71	$\frac{s+a}{s[(s+b)^2+\omega^2]}$	$\begin{cases} \frac{a}{b^2+\omega^2} - \frac{1}{\omega} + \sqrt{\frac{(a-b)^2+\omega^2}{b^2+\omega^2}}\,e^{-bt}\sin(\omega t+\phi); \\ \phi = \tan^{-1}\left(\frac{\omega}{b}\right) + \tan^{-1}\left(\frac{\omega}{a-b}\right) \end{cases}$
72	$\frac{s+a}{s^2[(s+b)^2+\omega^2]}$	$\begin{cases} \frac{1}{b^2+\omega^2}[1+at] - \frac{2ab}{(b^2+\omega^2)^2} + \frac{\sqrt{(a-b)^2+\omega^2}}{\omega(b^2+\omega^2)}e^{-bt}\sin(\omega t+\phi) \\ \phi = \tan^{-1}\left(\frac{\omega}{a-b}\right) + 2\tan^{-1}\left(\frac{\omega}{b}\right) \end{cases}$
73	$\frac{s+a}{(s+c)[(s+b)^2+\omega^2]}$	$\begin{cases} \frac{a-c}{(c-b)^2+\omega^2}e^{-ct} + \frac{1}{\omega}\sqrt{\frac{(a-b)^2+\omega^2}{(c-b)^2+\omega^2}}\,e^{-bt}\sin(\omega t+\phi) \\ \phi = \tan^{-1}\left(\frac{\omega}{a-b}\right) - \tan^{-1}\left(\frac{\omega}{c-b}\right) \end{cases}$
74	$\frac{s+a}{s(s+c)[(s+b)^2+\omega^2]}$	$\begin{cases} \frac{a}{c(b^2+\omega^2)} + \frac{(c-a)}{c[(b-c)^2+\omega^2]}e^{-ct} \\ -\frac{1}{\omega\sqrt{b^2+\omega^2}}\sqrt{\frac{(a-b)^2+\omega^2}{(b-c)^2+\omega^2}}\,e^{-bt}\sin(\omega t+\phi) \\ \phi = \tan^{-1}\left(\frac{\omega}{b}\right) + \tan^{-1}\left(\frac{\omega}{a-b}\right) - \tan^{-1}\left(\frac{\omega}{c-b}\right) \end{cases}$
75	$\frac{s+a}{s^2(s+b)^3}$	$\frac{a}{b^3}t + \frac{b-3a}{b^4} + \left[\frac{3a-b}{b^4} + \frac{a-b}{2b^2}t^2 + \frac{2a-b}{b^3}t\right]e^{-bt}$

	$F(s)$	$f(t)$
76	$\frac{s+a}{(s+c)(s+b)^3}$	$\frac{a-c}{(b-c)^3}e^{-ct}+\left[\frac{a-b}{2(c-b)}t^2+\frac{c-a}{(c-b)^2}t+\frac{a-c}{(c-b)^3}\right]e^{-bt}$
77	$\frac{s^2}{(s+a)(s+b)(s+c)}$	$\frac{a^2}{(b-a)(c-a)}e^{-at}+\frac{b^2}{(a-b)(c-b)}e^{-bt}+\frac{c^2}{(a-c)(b-c)}e^{-ct}$
78	$\frac{s^2}{(s+a)(s+b)^2}$	$\frac{a^2}{(b-a)^2}e^{-at}+\left[\frac{b^2}{(a-b)}t+\frac{b^2-2ab}{(a-b)^2}\right]e^{-bt}$
79	$\frac{s^2}{(s+a)^3}$	$\left[2-2at+\frac{a^2}{2}t^2\right]e^{-at}$
80	$\frac{s^2}{(s+a)(s^2+\omega^2)}$	$\frac{a^2}{(a^2+\omega^2)}e^{-at}-\frac{\omega}{\sqrt{a^2+\omega^2}}\sin(\omega t+\phi);\ \phi=\tan^{-1}\left(\frac{\omega}{a}\right)$
81	$\frac{s^2}{(s+a)^2(s^2+\omega^2)}$	$\left[\frac{a^2}{(a^2+\omega^2)}t-\frac{2a\omega^2}{(a^2+\omega^2)^2}\right]e^{-at}-\frac{\omega}{(a^2+\omega^2)}\sin(\omega t+\phi);$ $\phi=-2\tan^{-1}\left(\frac{\omega}{a}\right)$
82	$\frac{s^2}{(s+a)(s+b)(s^2+\omega^2)}$	$\frac{a^2}{(b-a)(a^2+\omega^2)}e^{-at}+\frac{b^2}{(a-b)(b^2+\omega^2)}e^{-bt}$ $-\frac{\omega}{\sqrt{(a^2+\omega^2)(b^2+\omega^2)}}\sin(\omega t+\phi);\ \phi=-\left[\tan^{-1}\left(\frac{\omega}{a}\right)+\tan^{-1}\left(\frac{\omega}{b}\right)\right]$
83	$\frac{s^2}{(s^2+a^2)(s^2+\omega^2)}$	$-\frac{a}{(\omega^2-a^2)}\sin(at)-\frac{\omega}{(a^2-\omega^2)}\sin(\omega t)$
84	$\frac{s^2}{(s^2+\omega^2)^2}$	$\frac{1}{2\omega}(\sin\omega t+\omega t\cos\omega t)$
85	$\frac{s^2}{(s+a)[(s+b)^2+\omega^2]}$	$\frac{a^2}{(a-b)^2+\omega^2}e^{-at}+\frac{1}{\omega}\sqrt{\frac{(b^2-\omega^2)^2+4b^2\omega^2}{(a-b)^2+\omega^2}}\,e^{-bt}\sin(\omega t+\phi)$ $\phi=\tan^{-1}\left(\frac{-2b\omega}{b^2-\omega^2}\right)-\tan^{-1}\left(\frac{\omega}{a-b}\right)$
86	$\frac{s^2}{(s+a)^2[(s+b)^2+\omega^2]}$	$\frac{a^2}{(a-b)^2+\omega^2}te^{-at}-2\left[\frac{a[(b-a)^2+\omega^2]+a^2(b-a)}{[(b-a)^2+\omega^2]^2}\right]e^{-at}$ $+\frac{\sqrt{(b^2-\omega^2)^2+4b^2\omega^2}}{\omega[(a-b)^2+\omega^2]}e^{-bt}\sin(\omega t+\phi)$ $\phi=\tan^{-1}\left(\frac{-2b\omega}{b^2-\omega^2}\right)-2\tan^{-1}\left(\frac{\omega}{a-b}\right)$
87	$\frac{s^2+a}{s^2(s+b)}$	$\frac{b^2+a}{b^2}e^{-bt}+\frac{a}{b}t-\frac{a}{b^2}$
88	$\frac{s^2+a}{s^3(s+b)}$	$\frac{a}{2b}t^2-\frac{a}{b^2}t+\frac{1}{b^3}[b^2+a-(a+b^2)e^{-bt}]$
89	$\frac{s^2+a}{s(s+b)(s+c)}$	$\frac{a}{bc}+\frac{(b^2+a)}{b(b-c)}e^{-bt}-\frac{(c^2+a)}{c(b-c)}e^{-ct}$
90	$\frac{s^2+a}{s^2(s+b)(s+c)}$	$\frac{b^2+a}{b^2(c-b)}e^{-bt}+\frac{c^2+a}{c^2(b-c)}e^{-ct}+\frac{a}{bc}t-\frac{a(b+c)}{b^2c^2}$

	$F(s)$	$f(t)$
91	$\dfrac{s^2+a}{(s+b)(s+c)(s+d)}$	$\dfrac{b^2+a}{(c-b)(d-b)}e^{-bt}+\dfrac{c^2+a}{(b-c)(d-c)}e^{-ct}+\dfrac{d^2+a}{(b-d)(c-d)}e^{-dt}$
92	$\dfrac{s^2+a}{s(s+b)(s+c)(s+d)}$	$\dfrac{a}{bcd}+\dfrac{b^2+a}{b(b-c)(d-b)}e^{-bt}+\dfrac{c^2+a}{c(b-c)(c-d)}e^{-ct}+\dfrac{d^2+a}{d(b-d)(d-c)}e^{-dt}$
93	$\dfrac{s^2+a}{s^2(s+b)(s+c)(s+d)}$	$\begin{cases}\dfrac{a}{bcd}t-\dfrac{a}{b^2c^2d^2}(bc+cd+db)+\dfrac{b^2+a}{b^2(b-c)(b-d)}e^{-bt}\\ +\dfrac{c^2+a}{c^2(c-b)(c-d)}e^{-ct}+\dfrac{d^2+a}{d^2(d-b)(d-c)}e^{-dt}\end{cases}$
94	$\dfrac{s^2+a}{(s^2+\omega^2)^2}$	$\dfrac{1}{2\omega^3}(a+\omega^2)\sin\omega t-\dfrac{1}{2\omega^2}(a-\omega^2)t\cos\omega t$
95	$\dfrac{s^2-\omega^2}{(s^2+\omega^2)^2}$	$t\cos\omega t$
96	$\dfrac{s^2+a}{s(s^2+\omega^2)^2}$	$\dfrac{a}{\omega^4}-\dfrac{(a-\omega^2)}{2\omega^3}t\sin\omega t-\dfrac{a}{\omega^4}\cos\omega t$
97	$\dfrac{s(s+a)}{(s+b)(s+c)^2}$	$\dfrac{b^2-ab}{(c-b)^2}e^{-bt}+\left[\dfrac{c^2-ac}{b-c}t+\dfrac{c^2-2bc+ab}{(b-c)^2}\right]e^{-ct}$
98	$\dfrac{s(s+a)}{(s+b)(s+c)(s+d)^2}$	$\begin{cases}\dfrac{b^2-ab}{(c-b)(d-b)^2}e^{-bt}+\dfrac{c^2-ac}{(b-c)(d-c)^2}e^{-ct}+\dfrac{d^2-ad}{(b-d)(c-d)}te^{-dt}\\ +\dfrac{a(bc-d^2)+d(db+dc-2bc)}{(b-d)^2(c-d)^2}e^{-dt}\end{cases}$
99	$\dfrac{s^2+a_1s+a_o}{s^2(s+b)}$	$\dfrac{b^2-a_1b+a_o}{b^2}e^{-bt}+\dfrac{a_o}{b}t+\dfrac{a_1b-a_o}{b^2}$
100	$\dfrac{s^2+a_1s+a_o}{s^3(s+b)}$	$\dfrac{a_1b-b^2-a_o}{b^3}e^{-bt}+\dfrac{a_o}{2b}t^2+\dfrac{a_1b-a_o}{b^2}t+\dfrac{b^2-a_1b+a_o}{b^3}$
101	$\dfrac{s^2+a_1s+a_o}{s(s+b)(s+c)}$	$\dfrac{a_o}{bc}+\dfrac{b^2-a_1b+a_o}{b(b-c)}e^{-bt}+\dfrac{c^2-a_1c+a_o}{c(c-b)}e^{-ct}$
102	$\dfrac{s^2+a_1s+a_o}{s^2(s+b)(s+c)}$	$\dfrac{a_o}{bc}t+\dfrac{a_1bc-a_o(b+c)}{b^2c^2}+\dfrac{b^2-a_1b+a_o}{b^2(c-b)}e^{-bt}+\dfrac{c^2-a_1c+a_o}{c^2(b-c)}e^{-ct}$
103	$\dfrac{s^2+a_1s+a_o}{(s+b)(s+c)(s+d)}$	$\dfrac{b^2-a_1b+a_o}{(c-b)(d-b)}e^{-bt}+\dfrac{c^2-a_1c+a_o}{(b-c)(d-c)}e^{-ct}+\dfrac{d^2-a_1d+a_o}{(b-d)(c-d)}e^{-dt}$
104	$\dfrac{s^2+a_1s+a_o}{s(s+b)(s+c)(s+d)}$	$\dfrac{a_o}{bcd}-\dfrac{b^2-a_1b+a_o}{b(c-b)(d-b)}e^{-bt}-\dfrac{c^2-a_1c+a_o}{c(b-c)(d-c)}e^{-ct}-\dfrac{d^2-a_1d+a_o}{d(b-d)(c-d)}e^{-dt}$
105	$\dfrac{s^2+a_1s+a_o}{s(s+b)^2}$	$\dfrac{a_o}{b^2}-\dfrac{b^2-a_1b+a_o}{b}te^{-bt}+\dfrac{b^2-a_o}{b^2}e^{-bt}$
106	$\dfrac{s^2+a_1s+a_o}{s^2(s+b)^2}$	$\dfrac{a_o}{b^2}t+\dfrac{a_1b-2a_o}{b^3}+\dfrac{b^2-a_1b+a_o}{b^2}t\varepsilon^{-bt}+\dfrac{2a_o-a_1b}{b^3}e^{-bt}$
107	$\dfrac{s^2+a_1s+a_o}{(s+b)(s+c)^2}$	$\dfrac{b^2-a_1b+a_o}{(c-b)^2}e^{-bt}+\dfrac{c^2-a_1c+a_o}{(b-c)}te^{-ct}+\dfrac{c^2-2bc+a_1b-a_o}{(b-c)^2}e^{-ct}$
108	$\dfrac{s^3}{(s+b)(s+c)(s+d)^2}$	$\begin{cases}\dfrac{b^3}{(b-c)(d-b)^2}e^{-bt}+\dfrac{c^3}{(c-b)(d-c)^2}e^{-ct}+\dfrac{d^3}{(d-b)(c-d)}te^{-dt}\\ +\dfrac{d^2[d^2-2d(b+c)+3bc]}{(b-d)^2(c-d)^2}e^{-dt}\end{cases}$

	$F(s)$	$f(t)$
109	$\dfrac{s^3}{(s+b)(s+c)(s+d)(s+f)^2}$	$\dfrac{b^3}{(b-c)(d-b)(f-b)^2}e^{-bt}+\dfrac{c^3}{(c-b)(d-c)(f-c)^2}e^{-ct}$ $+\dfrac{d^3}{(d-b)(c-d)(f-d)^2}e^{-dt}+\dfrac{f^3}{(f-b)(c-f)(d-f)}te^{-ft}$ $+\Bigg[\dfrac{3f^2}{(b-f)(c-f)(d-f)}$ $+\dfrac{f^3[(b-f)(c-f)+(b-f)(d-f)+(c-f)(d-f)]}{(b-f)^2(c-f)^2(d-f)^2}\Bigg]\varepsilon^{-dt}$
110	$\dfrac{s^3}{(s+b)^2(s+c)^2}$	$-\dfrac{b^3}{(c-b)^2}te^{-bt}+\dfrac{b^2(3c-b)}{(c-b)^3}e^{-bt}-\dfrac{c^3}{(b-c)^2}te^{-ct}+\dfrac{c^2(3b-c)}{(b-c)^3}e^{-ct}$
111	$\dfrac{s^3}{(s+d)(s+b)^2(s+c)^2}$	$-\dfrac{d^3}{(b-d)^2(c-d)^2}e^{-dt}+\dfrac{b^3}{(c-b)^2(b-d)}te^{-bt}$ $+\left[\dfrac{3b^2}{(c-b)^2(d-b)}+\dfrac{b^3(c+2d-3b)}{(c-b)^3(d-b)^2}\right]e^{-bt}+\dfrac{c^3}{(b-c)^2(c-d)}te^{-ct}$ $+\left[\dfrac{3c^2}{(b-c)^2(d-c)}+\dfrac{c^3(b+2d-3c)}{(b-c)^3(d-c)^2}\right]e^{-ct}$
112	$\dfrac{s^3}{(s+b)(s+c)(s^2+\omega^2)}$	$\dfrac{b^3}{(b-c)(b^2+\omega^2)}e^{-bt}+\dfrac{c^3}{(c-b)(c^2+\omega^2)}e^{-ct}$ $-\dfrac{\omega^2}{\sqrt{(b^2+\omega^2)(c^2+\omega^2)}}\sin(\omega t+\phi)$ $\phi=\tan^{-1}\left(\dfrac{c}{\omega}\right)-\tan^{-1}\left(\dfrac{\omega}{b}\right)$
113	$\dfrac{s^3}{(s+b)(s+c)(s+d)(s^2+\omega^2)}$	$\dfrac{b^3}{(b-c)(d-b)(b^2+\omega^2)}e^{-bt}+\dfrac{c^3}{(c-b)(d-c)(c^2+\omega^2)}e^{-ct}$ $+\dfrac{d^3}{(d-b)(c-d)(d^2+\omega^2)}e^{-dt}$ $-\dfrac{\omega^2}{\sqrt{(b^2+\omega^2)(c^2+\omega^2)(d^2+\omega^2)}}\cos(\omega t-\phi)$ $\phi=\tan^{-1}\left(\dfrac{\omega}{b}\right)+\tan^{-1}\left(\dfrac{\omega}{c}\right)+\tan^{-1}\left(\dfrac{\omega}{d}\right)$
114	$\dfrac{s^3}{(s+b)^2(s^2+\omega^2)}$	$-\dfrac{b^3}{b^2+\omega^2}te^{-bt}+\dfrac{b^2(b^2+3\omega^2)}{(b^2+\omega^2)^2}e^{-bt}-\dfrac{\omega^2}{(b^2+\omega^2)}\sin(\omega t+\phi)$ $\phi=\tan^{-1}\left(\dfrac{b}{\omega}\right)-\tan^{-1}\left(\dfrac{\omega}{b}\right)$
115	$\dfrac{s^3}{s^4+4\omega^4}$	$\cos(\omega t)\cosh(\omega t)$
116	$\dfrac{s^3}{s^4-\omega^4}$	$\frac{1}{2}[\cosh(\omega t)+\cos(\omega t)]$

	$F(s)$	$f(t)$
117	$\frac{s^3+a_2s^2+a_1s+a_o}{s^2(s+b)(s+c)}$	$\frac{a_o}{bc}t-\frac{a_o(b+c)-a_1bc}{b^2c^2}+\frac{-b^3+a_2b^2-a_1b+a_o}{b^2(c-b)}e^{-bt}+\frac{-c^3+a_2c^2-a_1c+a_o}{c^2(b-c)}e^{-ct}$
118	$\frac{s^3+a_2s^2+a_1s+a_o}{s(s+b)(s+c)(s+d)}$	$\frac{a_o}{bcd}-\frac{-b^3+a_2b^2-a_1b+a_o}{b(c-b)(d-b)}e^{-bt}-\frac{-c^3+a_2c^2-a_1c+a_o}{c(b-c)(d-c)}e^{-ct}-\frac{-d^3+a_2d^2-a_1d+a_o}{d(b-d)(c-d)}e^{-dt}$
119	$\frac{s^3+a_2s^2+a_1s+a_o}{s^2(s+b)(s+c)(s+d)}$	$\frac{a_o}{bcd}t+\left[\frac{a_1}{bcd}-\frac{a_o(bc+bd+cd)}{b^2c^2d^2}\right]+\frac{-b^3+a_2b^2-a_1b+a_o}{b^2(c-b)(d-b)}\varepsilon^{-bt}+\frac{-c^3+a_2c^2-a_1c+a_o}{c^2(b-c)(d-c)}e^{-ct}+\frac{-d^3+a_2d^2-a_1d+a_o}{d^2(b-d)(c-d)}e^{-dt}$
120	$\frac{s^3+a_2s^2+a_1s+a_o}{(s+b)(s+c)(s+d)(s+f)}$	$\frac{-b^3+a_2b^2-a_1b+a_o}{(c-b)(d-b)(f-b)}e^{-bt}+\frac{-c^3+a_2c^2-a_1c+a_o}{(b-c)(d-c)(f-c)}e^{-ct}+\frac{-d^3+a_2d^2-a_1d+a_o}{(b-d)(c-d)(f-d)}e^{-dt}+\frac{-f^3+a_2f^2-a_1f+a_o}{(b-f)(c-f)(d-f)}e^{-ft}$
121	$\frac{s^3+a_2s^2+a_1s+a_o}{s(s+b)(s+c)(s+d)(s+f)}$	$\frac{a_o}{bcdf}-\frac{-b^3+a_2b^2-a_1b+a_o}{b(c-b)(d-b)(f-b)}e^{-bt}-\frac{-c^3+a_2c^2-a_1c+a_o}{c(b-c)(d-c)(f-c)}e^{-ct}-\frac{-d^3+a_2d^2-a_1d+a_o}{d(b-d)(c-d)(f-d)}e^{-dt}-\frac{-f^3+a_2f^2-a_1f+a_o}{f(b-f)(c-f)(d-f)}e^{-ft}$
122	$\frac{s^3+a_2s^2+a_1s+a_o}{(s+b)(s+c)(s+d)(s+f)(s+g)}$	$\frac{-b^3+a_2b^2-a_1b+a_o}{(c-b)(d-b)(f-b)(g-b)}e^{-bt}+\frac{-c^3+a_2c^2-a_1c+a_o}{(b-c)(d-c)(f-c)(g-c)}e^{-ct}+\frac{-d^3+a_2d^2-a_1d+a_o}{(b-d)(c-d)(f-d)(g-d)}e^{-dt}+\frac{-f^3+a_2f^2-a_1f+a_o}{(b-f)(c-f)(d-f)(g-f)}e^{-ft}+\frac{-g^3+a_2g^2-a_1g+a_o}{(b-g)(c-g)(d-g)(f-g)}e^{-gt}$
123	$\frac{s^3+a_2s^2+a_1s+a_o}{(s+b)(s+c)(s+d)^2}$	$\frac{-b^3+a_2b^2-a_1b+a_o}{(c-b)(d-b)^2}e^{-bt}+\frac{-c^3+a_2c^2-a_1c+a_o}{(b-c)(d-c)^2}e^{-ct}+\frac{-d^3+a_2d^2-a_1d+a_o}{(b-d)(c-d)}te^{-dt}+\frac{a_o(2d-b-c)+a_1(bc-d^2)+a_2d(db+dc-2bc)+d^2(d^2-2db-2dc+3bc)}{(b-d)^2(c-d)^2}e^{-dt}$
124	$\frac{s^3+a_2s^2+a_1s+a_o}{s(s+b)(s+c)(s+d)^2}$	$\frac{a_o}{bcd^2}-\frac{-b^3+a_2b^2-a_1b+a_o}{b(c-b)(d-b)^2}e^{-bt}-\frac{-c^3+a_2c^2-a_1c+a_o}{c(b-c)(d-c)^2}e^{-ct}-\frac{-d^3+a_2d^2-a_1d+a_o}{d(b-d)(c-d)}te^{-dt}-\frac{3d^2-2a_2d+a_1}{d(b-d)(c-d)}e^{-dt}-\frac{(-d^3+a_2d^2-a_1d+a_o)[(b-d)(c-d)-d(b-d)-d(c-d)]}{d^2(b-d)^2(c-d)^2}e^{-dt}$

	$F(s)$	$f(t)$
125	$\frac{s^3+a_2s^2+a_1s+a_o}{(s+b)(s+c)(s+d)(s+f)^2}$	$\begin{cases} \frac{-b^3+a_2b^2-a_1b+a_o}{(c-b)(d-b)(f-b)^2}e^{-bt}+\frac{-c^3+a_2c^2-a_1c+a_o}{(b-c)(d-c)(f-c)^2}e^{-ct} \\ +\frac{-d^3+a_2d^2-a_1d+a_o}{(b-d)(c-d)(f-d)^2}e^{-dt}+\frac{-f^3+a_2f^2-a_1f+a_o}{(b-f)(c-f)(d-f)}te^{-ft} \\ +\frac{3f^2-2a_2f+a_1}{(b-f)(c-f)(d-f)}e^{-ft}-\frac{(-f^3+a_2f^2-a_1f+a_o)[(b-f)(c-f)+(b-f)(d-f)+(c-f)(d-f)]}{(b-f)^2(c-f)^2(d-f)^2}e^{-ft} \end{cases}$
126	$\frac{s}{(s-a)^{3/2}}$	$\frac{1}{\sqrt{\pi t}}e^{at}(1+2at)$
127	$\sqrt{s-a}-\sqrt{s-b}$	$\frac{1}{2\sqrt{\pi t^3}}(e^{bt}-e^{at})$
128	$\frac{1}{\sqrt{s}+a}$	$\frac{1}{\sqrt{\pi t}}-ae^{a^2t}\operatorname{erfc}(a\sqrt{t})$
129	$\frac{\sqrt{s}}{s-a^2}$	$\frac{1}{\sqrt{\pi t}}+ae^{a^2t}\operatorname{erf}(a\sqrt{t})$
130	$\frac{\sqrt{s}}{s+a^2}$	$\frac{1}{\sqrt{\pi t}}-\frac{2a}{\sqrt{\pi}}e^{-a^2t}\int_0^{a\sqrt{t}}e^{\lambda^2}\,d\lambda$
131	$\frac{1}{\sqrt{s}(s-a^2)}$	$\frac{1}{a}e^{a^2t}\operatorname{erf}(a\sqrt{t})$
132	$\frac{1}{\sqrt{s}(s+a^2)}$	$\frac{2}{a\sqrt{\pi}}e^{-a^2t}\int_0^{a\sqrt{t}}e^{\lambda^2}\,d\pi$
133	$\frac{b^2-a^2}{(s-a^2)(b+\sqrt{s})}$	$e^{a^2t}[b-a\operatorname{erf}(a\sqrt{t})]-be^{b^2t}\operatorname{erfc}(b\sqrt{t})$
134	$\frac{1}{\sqrt{s}(\sqrt{s}+a)}$	$e^{a^2t}\operatorname{erfc}(a\sqrt{t})$
135	$\frac{1}{(s+a)\sqrt{s+b}}$	$\frac{1}{\sqrt{b-a}}e^{-at}\operatorname{erf}(\sqrt{b-a}\sqrt{t})$
136	$\frac{b^2-a^2}{\sqrt{s}(s-a^2)(\sqrt{s}+b)}$	$e^{a^2t}\left[\frac{b}{a}\operatorname{erf}(a\sqrt{t})-1\right]+e^{b^2t}\operatorname{erfc}(b\sqrt{t})$
137	$\frac{(1-s)^n}{s^{n+(1/2)}}$	$\begin{cases} \frac{n!}{(2n)!\sqrt{\pi t}}H_{2n}(\sqrt{t}) \\ \left[H_n(t)=\text{Hermite polynomial}=e^{x^2}\frac{d^n}{dx^n}(e^{-x^2})\right] \end{cases}$
138	$\frac{(1-s)^n}{s^{n+(3/2)}}$	$-\frac{n!}{\sqrt{\pi}(2n+1)!}H_{2n+1}(\sqrt{t})$
139	$\frac{\sqrt{s+2a}}{\sqrt{s}}-1$	$\begin{cases} ae^{-at}[I_1(at)+I_o(at)] \\ [I_n(t)=j^{-n}J_n(jt)\text{ where }J_n\text{ is Bessel's function of the first kind}] \end{cases}$
140	$\frac{1}{\sqrt{s+a}\sqrt{s+b}}$	$e^{-(1/2)(a+b)t}I_o\left(\frac{a-b}{2}t\right)$

	$F(s)$	$f(t)$
141	$\frac{\Gamma(k)}{(s+a)^k(s+b)^k}(k\ge 0)$	$\sqrt{\pi}\left(\frac{t}{a-b}\right)^{k-(1/2)} e^{-(1/2)(a+b)t} I_{k-(1/2)}\left(\frac{a-b}{2}t\right)$
142	$\frac{1}{(s+a)^{1/2}(s+b)^{3/2}}$	$t e^{-(1/2)(a+b)t}\left[I_o\left(\frac{a-b}{2}t\right)+I_1\left(\frac{a-b}{2}t\right)\right]$
143	$\frac{\sqrt{s+2a}-\sqrt{s}}{\sqrt{s+2a}+\sqrt{s}}$	$\frac{1}{t}e^{-at}I_1(at)$
144	$\frac{(a-b)^k}{(\sqrt{s+a}+\sqrt{s+b})^{2k}} \quad (k>0)$	$\frac{k}{t}e^{-(1/2)(a+b)t}I_k\left(\frac{a-b}{2}t\right)$
145	$\frac{(\sqrt{s+a}+\sqrt{s})^{-2v}}{\sqrt{s}\sqrt{s+a}}$	$\frac{1}{a^v}e^{-(1/2)(at)}I_v\left(\frac{1}{2}at\right)$
146	$\frac{1}{\sqrt{s^2+a^2}}$	$J_o(at)$
147	$\frac{(\sqrt{s^2+a^2}-s)^v}{\sqrt{s^2+a^2}} \quad (v>-1)$	$a^v J_v(at)$
148	$\frac{1}{(s^2+a^2)^k} \quad (k>0)$	$\frac{\sqrt{\pi}}{\Gamma(k)}\left(\frac{t}{2a}\right)^{k-(1/2)} J_{k-(1/2)}(at)$
149	$(\sqrt{s^2+a^2}-s)^k \quad (k>0)$	$\frac{ka^k}{t}J_k(at)$
150	$\frac{(s-\sqrt{s^2-a^2})^v}{\sqrt{s^2-a^2}} \quad (v>-1)$	$a^v I_v(at)$
151	$\frac{1}{(s^2-a^2)^k} \quad (k>0)$	$\frac{\sqrt{\pi}}{\Gamma(k)}\left(\frac{t}{2a}\right)^{k-(1/2)} I_{k-(1/2)}(at)$
152	$\frac{1}{s\sqrt{s+1}}$	$\text{erf}(\sqrt{t})$; $\text{erf}(y) \triangleq$ the error function $= \frac{2}{\sqrt{\pi}}\int_o^y e^{-u^2}du$
153	$\frac{1}{\sqrt{s^2+a^2}}$	$J_o(at)$; Bessel function of 1st kind, zero order
154	$\frac{1}{\sqrt{s^2+a^2}+s}$	$\frac{J_1(at)}{at}$; J_1 is the Bessel function of 1st kind, 1st order
155	$\frac{1}{[\sqrt{s^2+a^2}+s]^N}$	$\frac{N}{a^N}\frac{J_N(at)}{t}$; $N=1,2,3,\cdots$, J_N is the Bessel function of 1st kind, Nth order
156	$\frac{1}{s[\sqrt{s^2+a^2}+s]^N}$	$\frac{N}{a^N}\int_o^t \frac{J_N(au)}{u}du$; $N=1,2,3,\cdots$, J_N is the Bessel function of 1st kind, Nth order
157	$\frac{1}{\sqrt{s^2+a^2}(\sqrt{s^2+a^2}+s)}$	$\frac{1}{a}J_1(at)$; J_1 is the Bessel function of 1st kind, 1st order
158	$\frac{1}{\sqrt{s^2+a^2}[\sqrt{s^2+a^2}+s]^N}$	$\frac{1}{a^N}J_N(at)$; $N=1,2,3,\cdots$, J_N is the Bessel function of 1st kind, Nth order
159	$\frac{1}{\sqrt{s^2-a^2}}$	$I_o(at)$; I_o is the modified Bessel function of 1st kind, zero order
160	$\frac{e^{-ks}}{s}$	$S_k(t)=\begin{cases}0 & \text{when } 0<t<k \\ 1 & \text{when } t>k\end{cases}$

	$F(s)$	$f(t)$
161	$\dfrac{e^{-ks}}{s^2}$	$\begin{cases} 0 & \text{when } 0<t<k \\ t-k & \text{when } t>k \end{cases}$
162	$\dfrac{e^{-ks}}{s^{\mu}} \quad (\mu>0)$	$\begin{cases} 0 & \text{when } 0<t<k \\ \dfrac{(t-k)^{\mu-1}}{\Gamma(\mu)} & \text{when } t>k \end{cases}$
163	$\dfrac{1-e^{-ks}}{s}$	$\begin{cases} 1 & \text{when } 0<t<k \\ 0 & \text{when } t>k \end{cases}$
164	$\dfrac{1}{s(1-e^{-ks})}=\dfrac{1+\coth\frac{1}{2}ks}{2s}$	$S(k,t)=\begin{cases} n & \text{when} \\ & (n-1)k<t<n \quad k(n=1,2,\cdots) \end{cases}$
165	$\dfrac{1}{s(e^{+ks}-a)}$	$S_k(t)=\begin{cases} 0 \quad \text{when } 0<t<k \\ 1+a+a^2+\cdots+a^{n-1} \\ \quad \text{when } nk<t<(n+1)k\ (n=1,2,\cdots) \end{cases}$
166	$\dfrac{1}{s}\tanh ks$	$\begin{cases} M(2k,t)=(-1)^{n-1} \\ \quad \text{when } 2k(n-1)<t<2nk \\ \quad (n=1,2,\cdots) \end{cases}$
167	$\dfrac{1}{s(1+e^{-ks})}$	$\begin{cases} \frac{1}{2}M(k,t)+\frac{1}{2}=\frac{1-(-1)^n}{2} \\ \quad \text{when } (n-1)k<t<nk \end{cases}$
168	$\dfrac{1}{s^2}\tanh ks$	$\begin{cases} H(2k,t) \quad [H(2k,t)=k+(r-k)(-1)^n \text{ where } t=2kn+r; \\ \quad 0\le r\le 2k;\ n=0,1,2,\cdots] \end{cases}$
169	$\dfrac{1}{s\sinh ks}$	$\begin{cases} 2S(2k,t+k)-2=2(n-1) \\ \quad \text{when } (2n-3)k<t<(2n-1)k \quad (t>0) \end{cases}$
170	$\dfrac{1}{s\cosh ks}$	$\begin{cases} M(2k,t+3k)+1=1+(-1)^n \\ \quad \text{when } (2n-3)k<t<(2n-1)k \quad (t>0) \end{cases}$
171	$\dfrac{1}{s}\coth ks$	$\begin{cases} 2S(2k,t)-1=2n-1 \\ \quad \text{when } 2k(n-1)<t<2kn \end{cases}$
172	$\dfrac{k}{s^2+k^2}\coth\dfrac{\pi s}{2k}$	$\lvert\sin kt\rvert$
173	$\dfrac{1}{(s^2+1)(1-e^{-\pi s})}$	$\begin{cases} \sin t & \text{when } (2n-2)\pi<t<(2n-1)\pi \\ 0 & \text{when } (2n-1)\pi<t<2n\pi \end{cases}$
174	$\dfrac{1}{s}e^{-k/s}$	$J_o(2\sqrt{kt})$
175	$\dfrac{1}{\sqrt{s}}e^{-k/s}$	$\dfrac{1}{\sqrt{\pi t}}\cos 2\sqrt{kt}$
176	$\dfrac{1}{\sqrt{s}}e^{k/s}$	$\dfrac{1}{\sqrt{\pi t}}\cosh 2\sqrt{kt}$
177	$\dfrac{1}{s^{3/2}}e^{-k/s}$	$\dfrac{1}{\sqrt{\pi k}}\sin 2\sqrt{kt}$
178	$\dfrac{1}{s^{3/2}}e^{k/s}$	$\dfrac{1}{\sqrt{\pi k}}\sinh 2\sqrt{kt}$

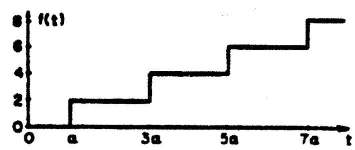
	$F(s)$	$f(t)$
179	$\frac{1}{s^{\mu}} e^{-k/s} \quad (\mu > 0)$	$\left(\frac{t}{k}\right)^{(\mu-1)/2} J_{\mu-1}(2\sqrt{kt})$
180	$\frac{1}{s^{\mu}} e^{k/s} \quad (\mu > 0)$	$\left(\frac{t}{k}\right)^{(\mu-1)/2} I_{\mu-1}(2\sqrt{kt})$
181	$e^{-k\sqrt{s}} \quad (k > 0)$	$\frac{k}{2\sqrt{\pi t^3}} \exp\left(-\frac{k^2}{4t}\right)$
182	$\frac{1}{s} e^{-k\sqrt{s}} \quad (k \geq 0)$	$\operatorname{erfc}\left(\frac{k}{2\sqrt{t}}\right)$
183	$\frac{1}{\sqrt{s}} e^{-k\sqrt{s}} \quad (k \geq 0)$	$\frac{1}{\sqrt{\pi t}} \exp\left(-\frac{k^2}{4t}\right)$
184	$s^{-3/2} e^{-k\sqrt{s}} \quad (k \geq 0)$	$2\sqrt{\frac{t}{\pi}} \exp\left(-\frac{k^2}{4t}\right) - k \operatorname{erfc}\left(\frac{k}{2\sqrt{t}}\right)$
185	$\frac{ae^{-k\sqrt{s}}}{s(a+\sqrt{s})} \quad (k \geq 0)$	$-e^{ak} e^{a^2 t} \operatorname{erfc}\left(a\sqrt{t} + \frac{k}{2\sqrt{t}}\right) + \operatorname{erfc}\left(\frac{k}{2\sqrt{t}}\right)$
186	$\frac{e^{-k\sqrt{s}}}{\sqrt{s}(a+\sqrt{s})} \quad (k \geq 0)$	$e^{ak} e^{a^2 t} \operatorname{erfc}\left(a\sqrt{t} + \frac{k}{2\sqrt{t}}\right)$
187	$\frac{e^{-k\sqrt{s(s+a)}}}{\sqrt{s(s+a)}}$	$\begin{cases} 0 & \text{when } 0 < t < k \\ e^{-(1/2)at} I_o(\frac{1}{2}a\sqrt{t^2-k^2}) & \text{when } t > k \end{cases}$
188	$\frac{e^{-k\sqrt{s^2+a^2}}}{\sqrt{(s^2+a^2)}}$	$\begin{cases} 0 & \text{when } 0 < t < k \\ J_o(a\sqrt{t^2-k^2}) & \text{when } t > k \end{cases}$
189	$\frac{e^{-k\sqrt{s^2-a^2}}}{\sqrt{(s^2-a^2)}}$	$\begin{cases} 0 & \text{when } 0 < t < k \\ I_o(a\sqrt{t^2-k^2}) & \text{when } t > k \end{cases}$
190	$\frac{e^{-k(\sqrt{s^2+a^2}-s)}}{\sqrt{(s^2+a^2)}} \quad (k \geq 0)$	$J_o(a\sqrt{t^2+2kt})$
191	$e^{-ks} - e^{-k\sqrt{s^2+a^2}}$	$\begin{cases} 0 & \text{when } 0 < t < k \\ \frac{ak}{\sqrt{t^2-k^2}} J_1(a\sqrt{t^2-k^2}) & \text{when } t > k \end{cases}$
192	$e^{-k\sqrt{s^2+a^2}} - e^{-ks}$	$\begin{cases} 0 & \text{when } 0 < t < k \\ \frac{ak}{\sqrt{t^2-k^2}} I_1(a\sqrt{t^2-k^2}) & \text{when } t > k \end{cases}$
193	$\frac{a^{\nu} e^{-k\sqrt{s^2-a^2}}}{\sqrt{(s^2+a^2)}\left(\sqrt{s^2+a^2}+s\right)^{\nu}}$ $(\nu > -1)$	$\begin{cases} 0 & \text{when } 0 < t < k \\ \left(\frac{t-k}{t+k}\right)^{(1/2)\nu} J_{\nu}(a\sqrt{t^2-k^2}) & \text{when } t > k \end{cases}$
194	$\frac{1}{s} \log s$	$\Gamma'(1) - \log t \quad [\Gamma'(1) = -0.5772]$
195	$\frac{1}{s^k} \log s \quad (k > 0)$	$t^{k-1}\left\{\frac{\Gamma'(k)}{[\Gamma(k)]^2} \frac{\log t}{\Gamma(k)}\right\}$
196	$\frac{\log s}{s-a} \quad (a > 0)$	$e^{at}[\log a - \operatorname{Ei}(-at)]$

	$F(s)$	$f(t)$
197	$\dfrac{\log s}{s^2+1}$	$\cos t\,\mathrm{Si}(t)-\sin t\,\mathrm{Ci}(t)$
198	$\dfrac{s\log s}{s^2+1}$	$-\sin t\,\mathrm{Si}(t)-\cos t\,\mathrm{Ci}(t)$
199	$\dfrac{1}{s}\log(1+ks)\quad(k>0)$	$-\mathrm{Ei}\left(-\dfrac{t}{k}\right)$
200	$\log\dfrac{s-a}{s-b}$	$\dfrac{1}{t}(e^{bt}-e^{at})$
201	$\dfrac{1}{s}\log(1+k^2s^2)$	$-2\mathrm{Ci}\left(\dfrac{t}{k}\right)$
202	$\dfrac{1}{s}\log(s^2+a^2)\quad(a>0)$	$2\log a-2\mathrm{Ci}(at)$
203	$\dfrac{1}{s^2}\log(s^2+a^2)\quad(a>0)$	$\dfrac{2}{a}[at\log a+\sin at-at\,\mathrm{Ci}(at)]$
204	$\log\dfrac{s^2+a^2}{s^2}$	$\dfrac{2}{t}(1-\cos at)$
205	$\log\dfrac{s^2-a^2}{s^2}$	$\dfrac{2}{t}(1-\cosh at)$
206	$\arctan\dfrac{k}{s}$	$\dfrac{1}{t}\sin kt$
207	$\dfrac{1}{s}\arctan\dfrac{k}{s}$	$\mathrm{Si}(kt)$
208	$e^{k^2s^2}\mathrm{erfc}(ks)\quad(k>0)$	$\dfrac{1}{k\sqrt{\pi}}\exp\left(-\dfrac{t^2}{4k^2}\right)$
209	$\dfrac{1}{s}e^{k^2s^2}\mathrm{erfc}(ks)\quad(k>0)$	$\mathrm{erf}\left(\dfrac{t}{2k}\right)$
210	$e^{ks}\mathrm{erfc}(\sqrt{ks})\quad(k>0)$	$\dfrac{\sqrt{k}}{\pi\sqrt{t(t+k)}}$
211	$\dfrac{1}{\sqrt{s}}\mathrm{erfc}(\sqrt{ks})$	$\begin{cases}0 & \text{when } 0<t<k\\ (\pi t)^{-1/2} & \text{when } t>k\end{cases}$
212	$\dfrac{1}{\sqrt{s}}e^{ks}\mathrm{erfc}(\sqrt{ks})\quad(k>0)$	$\dfrac{1}{\sqrt{\pi(t+k)}}$
213	$\mathrm{erf}\left(\dfrac{k}{\sqrt{s}}\right)$	$\dfrac{1}{\pi t}\sin(2k\sqrt{t})$
214	$\dfrac{1}{\sqrt{s}}e^{k^2/s}\mathrm{erfc}\left(\dfrac{k}{\sqrt{s}}\right)$	$\dfrac{1}{\sqrt{\pi t}}e^{-2k\sqrt{t}}$
215	$-e^{as}\mathrm{Ei}(-as)$	$\dfrac{1}{t+a};\ (a>0)$
216	$\dfrac{1}{a}+se^{as}\mathrm{Ei}(-as)$	$\dfrac{1}{(t+a)^2};\ (a>0)$
217	$\left[\dfrac{\pi}{2}-\mathrm{Si}(s)\right]\cos s+\mathrm{Ci}(s)\sin s$	$\dfrac{1}{t^2+1}$

	$F(s)$	$f(t)$	
218	$K_o(ks)$	$\begin{cases} 0 & \text{when } 0<t<k \\ (t^2-k^2)^{-1/2} & \text{when } t>k \end{cases}$	[$K_n(t)$ is Bessel function of the second kind of imaginary argument]
219	$K_o(k\sqrt{s})$	$\frac{1}{2t}\exp\left(-\frac{k^2}{4t}\right)$	
220	$\frac{1}{s}e^{ks}K_1(ks)$	$\frac{1}{k}\sqrt{t(t+2k)}$	
221	$\frac{1}{\sqrt{s}}K_1(k\sqrt{s})$	$\frac{1}{k}\exp\left(-\frac{k^2}{4t}\right)$	
222	$\frac{1}{\sqrt{s}}e^{k/s}K_o\left(\frac{k}{s}\right)$	$\frac{2}{\sqrt{\pi t}}K_o(2\sqrt{2kt})$	
223	$\pi e^{-ks}I_o(ks)$	$\begin{cases} [t(2k-t)]^{-1/2} & \text{when } 0<t<2k \\ 0 & \text{when } t>2k \end{cases}$	
224	$e^{-ks}I_1(ks)$	$\begin{cases} \frac{k-t}{\pi k\sqrt{t(2k-t)}} & \text{when } 0<t<2k \\ 0 & \text{when } t>2k \end{cases}$	

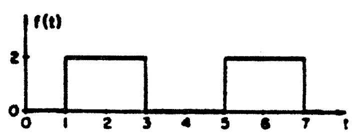
225 $\quad \frac{1}{s\sinh(as)}$

$$2\sum_{k=0}^{\infty} u[t-(2k+1)a]$$



226 $\quad \frac{1}{s\cosh s}$

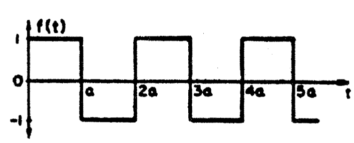
$$2\sum_{k=0}^{\infty}(-1)^k u(t-2k-1)$$



227 $\quad \frac{1}{s}\tanh\left(\frac{as}{2}\right)$

$$u(t)+2\sum_{k=1}^{\infty}(-1)^k u(t-ak)$$

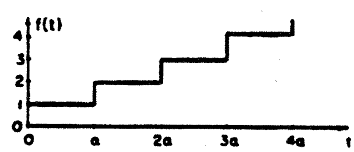
square wave



228 $\quad \frac{1}{2s}\left(1+\coth\frac{as}{2}\right)$

$$\sum_{k=0}^{\infty} u(t-ak)$$

stepped function

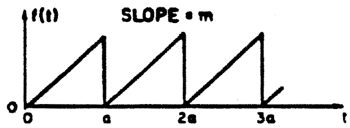


	$F(s)$	$f(t)$

229 $\quad \dfrac{m}{s^2} - \dfrac{ma}{2s}\left(\coth\dfrac{as}{2} - 1\right)$

$$mt - ma\sum_{k=1}^{\infty} u(t - ka)$$

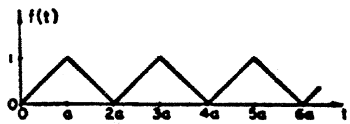
saw – tooth function



230 $\quad \dfrac{1}{s^2}\tanh\left(\dfrac{as}{2}\right)$

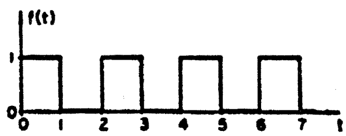
$$\frac{1}{a}\left[t + 2\sum_{k=1}^{\infty}(-1)^k (t - ka)\cdot u(t - ka)\right]$$

triangular wave



231 $\quad \dfrac{1}{s(1 + e^{-s})}$

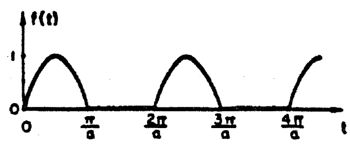
$$\sum_{k=0}^{\infty}(-1)^k u(t - k)$$



232 $\quad \dfrac{a}{(s^2 + a^2)(1 - e^{-\frac{\pi}{a}s})}$

$$\sum_{k=0}^{\infty}\left[\sin a\left(t - k\frac{\pi}{a}\right)\right]\cdot u\left(t - k\frac{\pi}{a}\right)$$

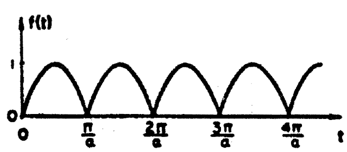
half – wave rectification of sine wave



233 $\quad \left[\dfrac{a}{(s^2 + a^2)}\right]\coth\left(\dfrac{\pi s}{2a}\right)$

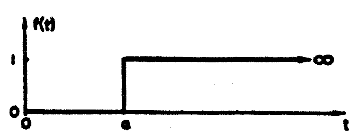
$$[\sin(at)]\cdot u(t) + 2\sum_{k=1}^{\infty}\left[\sin a\left(t - k\frac{\pi}{a}\right)\right]\cdot u\left(t - k\frac{\pi}{a}\right)$$

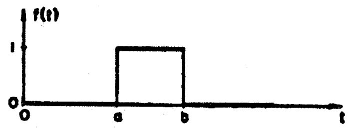
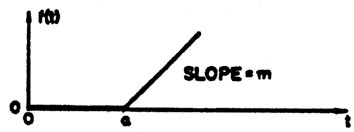
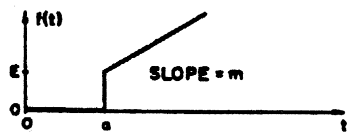
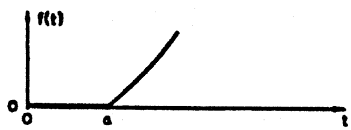
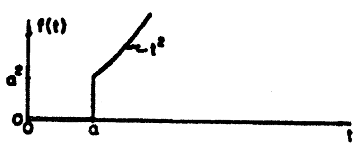
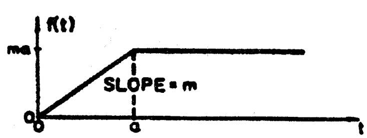
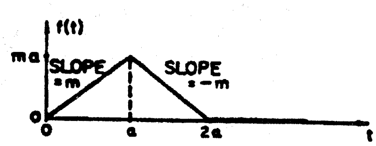
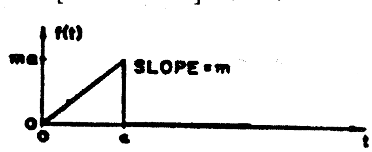
full – wave rectification of sine wave



234 $\quad \dfrac{1}{s}e^{-as}$

$$u(t - a)$$



	$F(s)$	$f(t)$
235	$\frac{1}{s}(e^{-as}-e^{-bs})$	$u(t-a)-u(t-b)$ 
236	$\frac{m}{s^2}e^{-as}$	$m\cdot(t-a)\cdot u(t-a)$ 
237	$\left[\frac{ma}{s}+\frac{m}{s^2}\right]e^{-as}$	$mt\cdot u(t-a)$ or $[ma+m(t-a)]\cdot u(t-a)$ 
238	$\frac{2}{s^3}e^{-as}$	$(t-a)^2\cdot u(t-a)$ 
239	$\left[\frac{2}{s^3}+\frac{2a}{s^2}+\frac{a^2}{s}\right]e^{-as}$	$t^2\cdot u(t-a)$ 
240	$\frac{m}{s^2}-\frac{m}{s^2}e^{-as}$	$mt\cdot u(t)-m(t-a)\cdot u(t-a)$ 
241	$\frac{m}{s^2}-\frac{2m}{s^2}e^{-as}+\frac{m}{s^2}e^{-2as}$	$mt-2m(t-a)\cdot u(t-a)+m(t-2a)\cdot u(t-2a)$ 
242	$\frac{m}{s^2}-\left(\frac{ma}{s}+\frac{m}{s^2}\right)e^{-as}$	$mt-[ma+m(t-a)]\cdot u(t-a)$ 

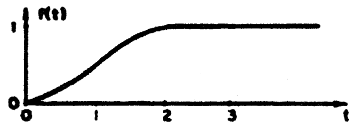
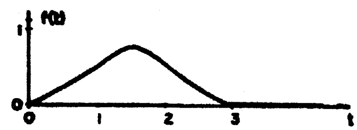
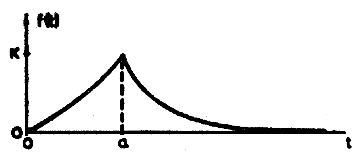
	$F(s)$	$f(t)$
243	$\dfrac{(1-e^{-s})^2}{s^3}$	$0.5t^2$ for $0 \le t < 1$ $1-0.5(t-2)^2$ for $0 \le t < 2$ 1 for $2 \le t$ 
244	$\left[\dfrac{(1-e^{-s})}{s}\right]^3$	$0.5t^2$ for $0 \le t < 1$ $0.75-(t-1.5)^2$ for $1 \le t < 2$ $0.5(t-3)^2$ for $2 \le t < 3$ 0 for $3 < t$ 
245	$\dfrac{b}{s(s-b)}+(e^{ba}-1)\left[\dfrac{1}{s+b}-\dfrac{s+\dfrac{b}{e^{ba}-1}}{s(s-b)}\right]e^{-as}$	$(e^{bt}-1)\cdot u(t)-(e^{bt}-1)\cdot u(t-a)+Ke^{-b(t-a)}\cdot u(t-a)$ where $K=(e^{ba}-1)$ 

TABLE 5.2 Properties of Laplace Transforms

	$F(s)$	$f(t)$
1	$\int_0^\infty e^{-st} f(t)\,dt$	$f(t)$
2	$AF(s) + BG(s)$	$Af(t) + Bg(t)$
3	$sF(s) - f(+0)$	$f'(t)$
4	$s^n F(s) - s^{n-1} f(+0) - s^{n-2} f^{(1)}(+0) - \cdots - f^{(n-1)}(+0)$	$f^{(n)}(t)$
5	$\frac{1}{s} F(s)$	$\int_0^t f(\tau)\,d\tau$
6	$\frac{1}{s^2} F(s)$	$\int_0^t \int_0^\tau f(\lambda)\,d\lambda\,d\tau$
7	$F_1(s) F_2(s)$	$\int_0^t f_1(t-\tau) f_2(\tau)\,d\tau = f_1 * f_2$
8	$-F'(s)$	$tf(t)$
9	$(-1)^n F^{(n)}(s)$	$t^n f(t)$
10	$\int_s^\infty F(x)\,dx$	$\frac{1}{t} f(t)$
11	$F(s-a)$	$e^{at} f(t)$
12	$e^{-bs} F(s)$	$f(t-b)$, where $f(t) = 0$; $t < 0$
13	$F(cs)$	$\frac{1}{c} f\left(\frac{t}{c}\right)$
14	$F(cs - b)$	$\frac{1}{c} e^{(bt)/c} f\left(\frac{t}{c}\right)$
15	$\dfrac{\int_0^a e^{-st} f(t)\,dt}{1 - e^{-as}}$	$f(t + a) = f(t)$ periodic signal
16	$\dfrac{\int_0^a e^{-st} f(t)\,dt}{1 + e^{-as}}$	$f(t + a) = -f(t)$
17	$\dfrac{F(s)}{1 - e^{-as}}$	$f_1(t)$, the half-wave rectification of $f(t)$ in No. 16.
18	$F(s) \coth \frac{as}{2}$	$f_2(t)$, the full-wave rectification of $f(t)$ in No. 16.
19	$\frac{p(s)}{q(s)}, q(s) = (s-a_1)(s-a_2)\cdots(s-a_m)$	$\sum_1^m \frac{p(a_n)}{q'(a_n)} e^{a_n t}$
20	$\frac{p(s)}{q(s)} = \frac{\phi(s)}{(s-a)^r}$	$e^{at} \sum_{n=1}^{r} \frac{\phi^{(r-n)}(a)}{(r-n)!} \frac{t^{n-1}}{(n-1)!} + \cdots$

Several additional transforms, especially those involving other Bessel functions, can be found in the following sources:

"Fourier Integrals for Practical Applications," G. A. Campbell and R. M. Foster, Van Nostrand, 1948. In these tables, only those entries containing the condition $0 < g$ or $k < g$, where g is our t, are Laplace transforms.

"Formulaire pour le calcul symbolique," N. W. McLachlan and P. Humbert, Gauthier–Villars, Paris, 1947.

"Tables of Integral Transforms," Bateman Manuscript Project, California Institute of Technology, A. Erdélyi and W. Magnus, Eds., McGraw-Hill, 1954; based on notes left by Harry Bateman.

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3. H. S. Carslaw and J. C. Jaeger, *Operational Methods in Applied Mathematics*, Dover Publications, Dover, NH, 1963.
4. W. R. LePage, *Complex Variables and the Laplace Transform for Engineers*, McGraw-Hill, New York, 1961.
5. R. E. Bolz and G. L. Turve, Eds., *CRC Handbook of Tables for Applied Engineering Science*, 2nd ed., CRC Press, Boca Raton, FL, 1973.
6. A. D. Poularikas and S. Seeley, *Signals and Systems*, corrected 2nd ed. Krieger Publishing Co., Melbourne, FL, 1994.